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Problem of Quotient Representation and Meromorphic Completion

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Introduction.

In the Symposium on "Application of the theory of functions of several complex variables to physics" held at Research Institute for Mathematical Sciences of Kyoto University in Feb. 1965, the following problem was presented by a physicist: *can any meromorphic function on a domain D in C^n be meromorphically continued to the envelope of holomorphy of D?*

The purpose of the present note is to solve this problem positively. For this purpose we shall discuss a meromorphic completion of any domain of $\boldsymbol{C}^n$ or, more generally, of a domain over a Stein manifold. We shall give the affirmative solution of the problem of quotient representation of meromorphic functions for any domain over a Stein manifold. As a result of this solution, we shall state an answer about the above problem.

The present note is a summary of the investigation about a meromorphic or holomorphic completion.

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1. Notations and definitions.

Let $\boldsymbol{C}^n$ denote the n-dimensional complex number space which is the direct product of n complex planes, each letter of $a, b, c, \ldots, w, z$ etc. denote a point of $\boldsymbol{C}^n$, $(z_1, \ldots, z_n)$ denote coordinates of a point z, and $\{|z_j - a_j| < r_j\}_{j=1}^n$ denote a *polydisc* which is the direct product of n discs $\{|z_j - a_j| < r_j\}$, $j = 1, \ldots, n$.

A domain (connected and open set) D of $\boldsymbol{C}^n$ is said to be a *Reinhardt domain* with center at the point c with coordinates $c_1, \ldots, c_n$ if its automorphisms consist of the n-parameter group $\{T(\theta_1, \ldots, \theta_n)\}$, each element of which is a biholomorphic mapping:

$$T(\theta_1, \ldots, \theta_n): \ z_j' = e^{i\theta_j}(z_j - c_j) + c_j, \ j = 1, \ldots, n,$$

where $0 \leqq \theta_j \leqq 2\pi$, $j = 1, \ldots, n$.

The Reinhardt domain D is said to be *proper* if the center c is an interior point of D. If along with each point $z^{(0)} \in D$ there belong to the domain D all the points z for which

$$|z_j - c_j| \leqq |z_j^{(0)} - c_j|, \ j = 1, \ldots, n,$$

the domain D is said to be a *complete* Reinhardt domain. Of course, a complete Reinhardt domain is proper. If D is a Reinhardt domain with center at the point c, then by means of a domain Δ in first closed quadrant $\{x_j \geqq 0\}_{j=1}^n$ of the n-dimensional real Euclid space $R^n(x_1, \ldots, x_n)$, D is expressed by the form

$$D = \{z \in C^n ; (|z_1 - c_1|, \ldots, |z_n - c_n|) \in \Delta\}.$$

Such a domain Δ we shall call a *domain of real expression* of D. Let D be a Reinhardt domain with center at the point c. If the domain Δ of real expression of D is mapped onto the (geometrically) convex domain Δ^* of the Euclid space $R^n(\xi_1, \ldots, \xi_n)$ by the transformation $\xi_j = \log|z_j - c_j|$, $j = 1, \ldots, n$, then D is said to have the *logarithmically convex* domain of real expression.

We shall define a meromorphic (or holomorphic) completion, an envelope of meromorphy (or holomorphy) and a domain of meromorphy (or holomorphy) for a domain over a complex analytic manifold as follows.

Let M be a complex analytic manifold. A *domain over* M is a pair (V, φ), where V is a connected and complex analytic manifold, and $\varphi: V \to M$ is a locally biholomorphic mapping. Consider domains (V, φ) and (V', φ') over M with a *mapping* λ of (V, φ) in (V', φ'), which is a holomorphic mapping $\lambda: V \to V'$ and satisfies $\varphi = \varphi' \circ \lambda$. Let f be a meromorphic (or holomorphic) function on V. A meromorphic (or holomorphic) function f' on V' with $f = f' \circ \lambda$ is called a *meromorphic* (or *holomorphic*) *continuation* of f to (λ, V', φ') or, briefly, to V'. Let $\mathfrak{F}$ be a family of meromorphic (or holomorphic) functions on V. If any meromorphic (or holomorphic) function of $\mathfrak{F}$ has a meromorphic (or holomorphic) continuation to V', then (λ, V', φ') or, briefly, (V', φ') is called a *meromorphic* (or *holomorphic*) *completion of* (V, φ) *with respect to the family* $\mathfrak{F}$. A meromorphic (or holomorphic) completion $(\tilde{\lambda}_{\mathfrak{F}}, \tilde{V}_{\mathfrak{F}}, \tilde{\varphi}_{\mathfrak{F}})$ or, briefly, $(\tilde{V}_{\mathfrak{F}}, \tilde{\varphi}_{\mathfrak{F}})$ of (V, φ) with respect to $\mathfrak{F}$ is called an *envelope of meromorphy* (or *holomorphy*) *of* (V, φ) *with respect to the family* $\mathfrak{F}$ if the following condition is satisfied: if (λ', V', φ') is another meromorphic (or holomorphic) completion of (V, φ) with respect to $\mathfrak{F}$, then there exists a mapping ψ of (V', φ') in $(\tilde{V}_{\mathfrak{F}}, \tilde{\varphi}_{\mathfrak{F}})$ with $\tilde{\lambda}_{\mathfrak{F}} = \psi \circ \lambda'$ such that $(\psi, \tilde{V}_{\mathfrak{F}}, \tilde{\varphi}_{\mathfrak{F}})$ is a meromorphic (or holomorphic) completion of (V', φ') with respect to the family $\mathfrak{F}'$ which consists of meromorphic (or holomorphic) continuation of all meromorphic (or holomorphic) functions of $\mathfrak{F}$ to V'.

If $\mathfrak{F}$ is the family of all meromorphic (or holomorphic) functions on V, then a meromorphic (or holomorphic) completion of (V, φ) with respect to $\mathfrak{F}$ and an envelope of meromorphy (or holomorphy) of (V, φ) with respect to $\mathfrak{F}$ are called simply a *meromorphic* (or *holomorphic*) *completion* of (V, φ) and an *envelope of meromorphy* (or *holomorphy*) of (V, φ) respectively.

By the same method as Malgrange [5], who proved the existence and the uniqueness of the envelope of holomorphy, we can prove those of the envelope of meromorphy.

The envelope $(\widetilde{\lambda}_f, \widetilde{V}_f, \widetilde{\varphi}_f)$ of meromorphy (or holomorphy) of (V, φ) with respect to the family consisting of only a function f which is meromorphic (or holomorphic) on V, is called the *domain of meromorphy* (or *holomorphy*) *of* f. A domain (V, φ) over M is called a *domain of meromorphy* (or *holomorphy*) if it is a domain of meromorphy (or holomorphy) of a meromorphic (or holomorphic) function on V.

2. Meromorphic completion of a Reinhardt domain.

In what follows, we deal with proper Reinhardt domains with center at the origin.

First of all we will discuss the meromorphic continuation concerning meromorphic functions in such a domain of the space C^n. For the case of $n = 2$, Thullen [8] has discussed in detail such a continuation. For the case of n complex variables, we have the another proof about theorems on meromorphic completion by the only use of the continuation theorem[1)] of Levi-Kneser.

By a remark of Okuda-Sakai [6], this theorem is equivalent to the following

Continuation Theorem. *If $f(z)$ is meromorphic (or holomorphic) in a neighbourhood of the union of the sets* $\{|z_1| = 1, z_2 = 0, \dots, z_{n-1} = 0, |z_n| \leqq 1\}$ *and* $\{|z_1| \leqq 1, z_2 = 0, \dots, z_{n-1} = 0, z_n = 0\}$, *then $f(z)$ can be meromorphically (or holomorphically) continued also to a neighbourhood of the set* $\{|z_1| \leqq 1, z_2 = 0, \dots, z_{n-1} = 0, |z_n| \leqq 1\}$.

Using the above continuation theorem, we have the following two lemmas.

Lemma 1. *If $f(z)$ is meromorphic (or holomorphic) in the domains* $\{a_1 < |z_1| < b_1, |z_2| < b_2, \dots, |z_n| < b_n\}$ *and* $\{|z_1| < b'_1, |z_2| < b'_2, \dots, |z_n| < b'_n\}$, *then $f(z)$ can be meromorphically (or holomorphically) continued to the polydisc* $\{|z_j| < b_j\}_{j=1}^n$, *where a_1, b_j and b'_j $(j = 1, \dots, n)$ are real numbers satisfying the condition* $0 < a_1 < b'_1 < b_1$ *and* $0 < b'_j < b_j$ $(j = 2, \dots, n)$.

Lemma 2. *If $f(z)$ is meromorphic (or holomorphic) in the domains* $\{|z_1| < b_1, a_2 < |z_2| < b_2, \dots, a_n < |z_n| < b_n\}$ *and* $\{|z_1| < b'_1, |z_2| < b'_2, \dots, |z_n| < b'_n\}$, *then $f(z)$ can be meromorphically (or holomorphically) continued to the polydisc* $\{|z_j| < b_j\}_{j=1}^n$, *where a_j $(j = 2, \dots, n)$, b_k and b'_k $(k = 1, \dots, n)$ are real numbers satisfying the condition* $0 < b'_1 < b_1$ *and* $0 < a_j < b'_j < b_j$ $(j = 2, \dots, n)$.

1) This is often called the "continuity theorem" (*Kontinuitätssatz*).

By the alternative use of these lemmas, we obtain easily the following theorem which is called the expansion theorem of H. Cartan for the case of holomorphic functions.

THEOREM 1. *If $f(z)$ is meromorphic (or holomorphic) in a proper Reinhardt domain D, then $f(z)$ can be meromorphically (or holomorphically) continued to the least complete Reinhardt domain containing D.*

Let $f(w_1, \ldots, w_n, z)$ be meromorphic (or holomorphic) in a neighbourhood of the set $\{w=(w_1, \ldots, w_n) \in D, z=c\}$, where D is a domain in the space $C^n(w)$. For each point $w^{(0)}$ in D, let $R(w^{(0)})$ denote the supremum of the set of radii r such that f is meromorphic (or holomorphic) in (w, z) in a neighbourhood of the set $\{w=w^{(0)}, |z-c|<r\}$. Then $R(w^{(0)})$ is called the *radius of meromorphy* (or *holomorphy*) of f at a point $w^{(0)}$ with center at the point c.

In [6], we have proved that the above continuation theorem is equivalent to the following assertion.

If $R(w_1, \ldots, w_n)$ is the radius of meromorphy (or holomorphy) of a meromorphic (or holomorphic) function $f(w_1, \ldots, w_n, z)$ at a point w, then for a fixed point $(w_2^{(0)}, \ldots, w_n^{(0)})$ $\log R(w_1, w_2^{(0)}, \ldots, w_n^{(0)})$ is superharmonic in w_1.

Using this result and Theorem 1, we obtain the following

THEOREM 2. *If $f(z)$ is meromorphic (or holomorphic) in a proper Reinhardt domain D, then $f(z)$ can be meromorphically (or holomorphically) continued to the least complete Reinhardt domain containing D which has the logarithmically convex domain of real expression.*

Consequently, it is concluded that the envelope of meromorphy of a proper Reinhardt domain D coincides with the envelope of holomorphy of D.

3. Meromorphic completion of a domain (D, φ).

A complex analytic manifold M is called to be of *weak* (or *strong*) *Poincaré type* if for any meromorphic function f on M there exist holomorphic functions g and h on M such that $f=h/g$ on M (or, in addition to this, g and h are coprime at each point of M). From Hitotumatu-Kôta [3] any Stein manifold is of weak Poincaré type. It is easy to see that for any domain (D, φ) over a Stein manifold, D is of weak Poincaré type if and only if the envelope of holomorphy of (D, φ) coincides with the envelope of meromorphy of (D, φ).

Using Lemma 1 and a result of Docquier-Grauert [2], we have the following

THEOREM 3. *Let (D, φ) be a domain over a Stein manifold, $\mathfrak{F}$ be a family of meromorphic functions on D and $(\tilde{\lambda}_{\mathfrak{F}}, \tilde{D}_{\mathfrak{F}}, \tilde{\varphi}_{\mathfrak{F}})$ be the envelope of meromorphy of (D, φ) with respect to $\mathfrak{F}$. Then $\tilde{D}_{\mathfrak{F}}$ is a Stein manifold.*

From Theorem 3 and Hitotumatu-Kôta [3], we have the following

THEOREM 4. *Let (D, φ) be a domain over a Stein manifold. Then D is of weak Poincaré type.*

As a immediate result of Theorem 4, we obtain

THEOREM 5. *Let (D, φ) be a domain over a Stein manifold. Then the envelope of holomorphy of (D, φ) coincides with the envelope of meromorphy of (D, φ). Especially any meromorphic function on D can be meromorphically continued to the envelope of holomorphy of (D, φ).*

Consequently, from Theorem 3, [1] and [2] we obtain

THEOREM 6. *Let (D, φ) be a domain over a Stein manifold. Then the following four conditions are equivalent:*

1). *(D, φ) is an envelope of meromorphy with respect to a family of meromorphic functions.*

2). *(D, φ) is a domain of meromorphy.*

3). *(D, φ) is a domain of holomorphy.*

4). *D is holomorphically convex.*

Full particulars will soon be reported in Kajiwara-Sakai [4] and Sakai [7].

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