

Lebesgue-Stieltjes Integrability of x^n with Respect to Unbounded Monotone Functions

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1. Introduction.

B. Sz.-Nagy [1] showed the following theorem.

Theorem A. If $0 < \gamma \leq 1$, $f(x)$ decreases on $(0, \pi)$, $f(\pi - 0) > -\infty$ and $xf(x) \in L(0, \pi)$, then $x^{\gamma-1}f(x) \in L(0, \pi)$ if and only if $\sum n^{-\gamma} |b_n| < \infty$ where $b_n = 2\pi^{-1} \int_0^\pi f(x) \sin nx \, dx$.

It is easy to check that the statement of Theorem A for $1 < \gamma < 2$ is still true. But when $\gamma = 0$, the theorem fails; as an example we may take $f(x) = -x$, then $x^{-1}f(x) \in L(0, \pi)$, $|b_n| = 2n^{-1}$ and $\sum |b_n| = \infty$. Recently as a replacement for the case $\gamma = 0$ of Theorem A, R. P. Boas [2] showed

Theorem B. If $f(x)$ decreases on $(0, \pi)$ and $\int_{(0, \pi)} x^2 |df(x)| < \infty$, then $f(x)$ is bounded if and only if $n^{-1} \sum_{k=1}^n kb_k = O(1)$ where $b_k = 2\pi^{-1} \int_0^\pi f(x) \sin kx \, dx$.

Moreover he [2] gave the following two theorems.

Theorem B'. If $f(x)$ decreases on $(0, \pi)$ and $\int_{(0, \pi)} x^2 |df(x)| < \infty$, then $f(x)$ is bounded if and only if $n^{-1} \sum_{k=1}^n a_k = O(1)$ where $a_k = -2\pi^{-1} \int_{(0, \pi)} (1 - \cos kx) df(x)$.

Theorem C. If $g(x) \geq 0$ on $(0, \pi)$ and $x^3g(x) \in L(0, \pi)$, then $xg(x) \in L(0, \pi)$ if and only if $n^{-1} \sum_{k=1}^n k^{-1}b_k = O(1)$ where $b_k = -2\pi^{-1} \int_0^\pi (kx - \sin kx) g(x) \, dx$.

It is evident by Nagy's lemma ([1], p. 119) that Theorem B' is equivalent to Theorem B. The aim of this paper is to give a generalization of these results being due to Boas.

2. Theorems.

Theorem 1. Let m be a non-negative integer and $f(x)$ be a function of bound-

ded variation on $[\varepsilon, \pi]$ for every $\varepsilon > 0$. (i) If

$$(2.1) \quad \int_{(0, \pi)} x^{2m} |df(x)| < \infty,$$

then we have

$$(2.2) \quad \frac{1}{n} \sum_{k=1}^n \frac{|a_k|}{k^{2m}} = O(1),$$

where

$$(2.3) \quad a_k = \frac{2}{\pi} \int_{(0, \pi)} \left\{ \cos kx - \sum_{j=0}^m (-1)^j \frac{(kx)^{2j}}{(2j)!} \right\} df(x) \quad k = 1, 2, \dots,$$

(ii) If

$$(2.4) \quad \int_{(0, \pi)} x^{2m+1} |df(x)| < \infty,$$

then we have

$$(2.5) \quad \frac{1}{n} \sum_{k=1}^n \frac{|b_k|}{k^{2m+1}} = O(1),$$

where

$$(2.6) \quad b_k = \frac{2}{\pi} \int_{(0, \pi)} \left\{ \sin kx - \sum_{j=0}^m (-1)^j \frac{(kx)^{2j+1}}{(2j+1)!} \right\} df(x) \quad k = 1, 2, \dots,$$

Theorem 2. Let m be a non-negative integer and $f(x)$ decrease on $(0, \pi)$.

(i) If

$$(2.7) \quad \int_{(0, \pi)} x^{2m+2} |df(x)| < \infty$$

and

$$(2.8) \quad \frac{1}{n} \sum_{k=1}^n \frac{a_k}{k^{2m}} = O(1),$$

where a_k 's are defined by (2.3), then we have (2.1). (ii) If

$$(2.9) \quad \int_{(0, \pi)} x^{2m+3} |df(x)| < \infty$$

and

$$(2.10) \quad \frac{1}{n} \sum_{k=1}^n \frac{b_k}{k^{2m+1}} = O(1),$$

where b_k 's are defined by (2.6), then we have (2.4).

The cases $m = 0$ in Theorem 1 (i) and Theorem 2 (i) yield Theorem B'. Similarly the cases $m = 0$ in Theorem 1 (ii) and Theorem 2 (ii) yield Theorem C.

Now we prove Theorem 1 (i) and Theorem 2 (i). The proof for Theorem 1 (ii) and Theorem 2 (ii) will be proceeded quite similarly.

Proof of Theorem 1 (i).

By (2.1) and (2.3), we have

$$\begin{aligned} |a_k| &\leq \frac{2}{\pi} \int_{(0, \pi)} 2 \frac{(kx)^{2m}}{(2m)!} |df(x)| \\ &= k^{2m} \frac{4}{\pi} \frac{1}{(2m)!} \int_{(0, \pi)} x^{2m} |df(x)| = C k^{2m}, \quad k = 1, 2, \dots, \end{aligned}$$

Hence

$$0 \leq \frac{1}{n} \sum_{k=1}^n \frac{|a_k|}{k^{2m}} \leq C, \quad n = 1, 2, \dots.$$

Proof of Theorem 2 (i).

First of all let $m = 0$. Then

$$a_k = \frac{2}{\pi} \int_{(0, \pi)} (1 - \cos kx) |df(x)| \geq 0, \quad k = 1, 2, \dots,$$

$$0 \leq \frac{1}{n} \sum_{k=1}^n a_k \leq C, \quad n = 1, 2, \dots,$$

and

$$\begin{aligned} \frac{\pi}{2} \frac{1}{n} \sum_{k=1}^n a_k &= \frac{1}{n} \sum_{k=1}^n \int_{(0, \pi)} (1 - \cos kx) |df(x)| \\ &= \int_{(0, \pi)} \left\{ 1 - \frac{1}{n} \frac{\sin(n + \frac{1}{2})x - \sin \frac{x}{2}}{2 \sin \frac{x}{2}} \right\} |df(x)|. \end{aligned}$$

Letting $n \rightarrow \infty$, we have $\frac{\pi}{2} C \geq \int_{(0, \pi)} |df(x)|$ by Fatou's lemma.

Secondly let $m \geq 1$. Then

$$\begin{aligned} \left| \frac{1}{n} \sum_{k=1}^n \frac{a_k}{k^{2m}} \right| &\leq C, \quad n = 1, 2, \dots, \\ \frac{\pi}{2} \frac{1}{n} \sum_{k=1}^n \frac{a_k}{k^{2m}} &= \frac{1}{n} \sum_{k=1}^n \frac{1}{k^{2m}} \int_{(0, \pi)} \left\{ \cos kx - \sum_{j=0}^{m-1} (-1)^j \frac{(kx)^{2j}}{(2j)!} \right\} df(x) \\ &= \int_{(0, \pi)} \left\{ \frac{(-1)^m}{(2m)!} x^{2m} - \frac{1}{n} \sum_{k=1}^n \frac{1}{k^{2m}} \left(\cos kx - \sum_{j=0}^{m-1} (-1)^j \frac{(kx)^{2j}}{(2j)!} \right) \right\} |df(x)|. \end{aligned}$$

Therefore

$$\begin{aligned} &(-1)^m \frac{\pi}{2} \frac{1}{n} \sum_{k=1}^n \frac{a_k}{k^{2m}} \\ &= \int_{(0, \pi)} \left\{ \frac{x^{2m}}{(2m)!} - \frac{1}{n} \sum_{k=1}^n \frac{(-1)^m}{k^{2m}} \left(\cos kx - \sum_{j=0}^{m-1} (-1)^j \frac{(kx)^{2j}}{(2j)!} \right) \right\} |df(x)|. \end{aligned}$$

And we know

$$\begin{aligned} &\frac{1}{n} \sum_{k=1}^n \frac{(-1)^m}{k^{2m}} \left(\cos kx - \sum_{j=0}^{m-1} (-1)^j \frac{(kx)^{2j}}{(2j)!} \right) \\ &= \frac{1}{n} (-1)^m \sum_{k=1}^n \frac{\cos kx}{k^{2m}} - \sum_{j=0}^{m-1} \frac{(-1)^{m+j} x^{2j}}{(2j)!} \frac{1}{n} \sum_{k=1}^n \frac{1}{k^{2(m-j)}} \rightarrow 0 \quad (n \rightarrow \infty) \end{aligned}$$

and

$$\frac{x^{2m}}{(2m)!} - \frac{1}{n} \sum_{k=1}^n \frac{(-1)^m}{k^{2m}} \left(\cos kx - \sum_{j=0}^{m-1} (-1)^j \frac{(kx)^{2j}}{(2j)!} \right) \geq 0$$

because

$$\left| \cos kx - \sum_{j=0}^{m-1} (-1)^j \frac{(kx)^{2j}}{(2j)!} \right| \leq \frac{(kx)^{2m}}{(2m)!}.$$

Hence by Fatou's lemma

$$\frac{\pi}{2} C \geq \frac{1}{(2m)!} \int_{(0, \pi)} x^{2m} |df(x)|.$$

References

- [1] B. Sz.-NAGY, Séries et intégrales de Fourier des fonctions monotones non bornées, Acta Sci. Math. 13 (1949), 118-135.
- [2] R. P. BOAS, Jr., Integrability of non-negative trigonometric series II, Tôhoku Math. Journ. 16 (1964), 368-373.