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$|C,1|$- summability of Fourier series with some gaps

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1. W. C. Randels [1] and M. Kiyohara [2] obtained the following

Theorem A. Let $f(x)$ be Lebesgue integrable in the interval $(-\pi, \pi)$ with period 2π. If at every point y on the closed interval $[-\pi, \pi]$, there exit a function $g_y(x)$ and a $\delta=\delta_y>0$ such that (i) $g_y(x)=f(x)$ *for* $|x-y|<\delta$, *and* (ii) *the Fourier series of $g_y(x)$ is $|C, \alpha|$-summable for an α $(0<\alpha\leqq 1)$, then the Fourier series of $f(x)$ is $|C, \alpha|$-summable.*

This is analogous to a theorem of the absolute convergence proved by N. Wiener [3], and a key point of the proof of this theorem is the following

Theorem B. If the Fourier series $g_y(x)$ is $|C, \alpha|$-summable $(0<\alpha\leqq 1)$ at every point x, then the Fourier series of $g_y(x)\cdot h(x-y)$ is also $|C, \alpha|$-summable at every point x, where $h(x)$ is an even and periodic function with period 2π, and defined by

$$h(x)=\begin{cases} A(x-\delta)^3+B(x-\delta)^2 & \text{for } \frac{\delta}{2}\leqq x\leqq\delta \\ 1 & \text{for } 0\leqq x\leqq\frac{\delta}{2} \\ 0 & \text{for } \delta\leqq x\leqq\pi \end{cases} \tag{1.1}$$

$$h(\delta/2)=1, \qquad h'(\delta/2)=0.$$

The above function $h(x)$ is exactly determined, i. e. $A=16\delta^{-3}$ and $B=12\delta^{-2}$. Though we have by (1.1)

$$h(x-y)\cdot g_y(x)=g_y(x)=f(x) \qquad \text{for} \qquad |x-y|\leqq\frac{\delta}{2}$$

we do'nt know whether the Fourier series of $f(x)$ is $|C, \alpha|$-summable in the interval $(y-\frac{\delta}{2}, y+\frac{\delta}{2})$ or not, under the $|C, \alpha|$-summability of the Fourier series of $g_y(x)$.

With regard to this problem the following theorem will be established.

Theorem. Let the Fourier series of $f(x)$ *and* $g(x)$ *be, respectively,*

$$(1.2)\qquad f(x) \sim \sum_p c_p e^{ipx}, \quad g(x) \sim \sum_p \gamma_p e^{ipx},$$

and let the former be a gap series satisfying the following gap conditions

$$(1.3)\qquad c_p = 0 \qquad \text{for} \qquad p \neq n_k \qquad k = 0, \pm 1, \pm 2, \cdots\cdots$$

where $\{n_k;\ k = 0, 1, 2, \cdots\cdots\}$ *is a nondecreasing sequence of integral numbers such that* (i)

$$(1.4)\qquad n_0 = 0, \qquad n_{-k} = -n_k \qquad k = 0, 1, 2, \cdots\cdots$$

and (ii) *the following conditions* (1.5) *are satisfied,*

$$(1.5)\qquad (a)\quad \frac{n_{k+1} - n_k}{n_k} \leqq C < \infty \qquad k = \pm 1, \pm 2, \cdots\cdots$$

$$(b)\quad \sum_1^\infty \frac{k}{n_k} < \infty$$

$$(c)\quad \sum_1^\infty \frac{n_k}{(n_{k+1} - n_k)^2} < \infty$$

If $g(x) = f(x)$ *in some interval* $(-\delta, \delta)$, *then from the* $|C, 1|$ *-summability of the Fourier series of* $g(x)$ *at every point* x, *the* $|C, 1|$*-summability of the Fourier series of* $f(x)$ *at every point* x *in the interval* $(-\delta, \delta)$, *follows.*

Remark 1. The case for $|C, \alpha > 1|$ - summability of our theorem follows immediately from the well known theorem of L. S. Bosanquet [4].

Remark 2. The sequence $\{k^4\}$ satisfies (1.5) (a), (b), (c). On the other hand let $\{n_k\}$ of our theorem satisfy the following conditions (1.6) in place of (1.5), i. e. *there exists a constant* K *such that if* $k \geqq K$, *then for any positive integer* l, *both*

$$(1.6)\qquad \begin{aligned} & l^4 < n_k < n_{k+1} < (l+1)^4 \\ & l^4 < n_k < (l+1)^4 < n_{k+1} < (l+2)^4 \end{aligned}$$

do not happen.

For this n_k let l_k^4 be the nearest integer from n_k in the set $\{0^4, 1^4, 2^4, \cdots\}$, and we define a new sequence $\{m_j\}$ of integers:

$$m_j = \begin{cases} j^4 & \text{for } j \neq l_k \\ n_k & \text{for } j = l_k \end{cases}$$

$$m_{-j} = -m_j \qquad k = 0, 1, 2, \cdots$$

Now we consider the trigonometric series $\sum_{-\infty}^{\infty} c_{m_j} e^{im_j x}$, which is the Fourier series of $f(x)$ by reason of $c_{m_j} = 0$ for $m_j \neq n_k$, and $\{m_j\}$ satisfies (1.5) (a), (b), (c). Thus for a gap series with gaps bigger than (1.5), if it does not satisfy both of

(1.6), then our theorem is applied.

2. We must first prove a few lemmas.

Lemma 1. If

$$h(x) \sim \sum_n d_n e^{inx}, \quad S_M(x) = \sum_{|n| \leqq M} d_n e^{inx}$$

are the Fourier series and its M-th partial sum of $h(x)$, then

$$(2.1) \qquad |S_M(x)| \leqq \begin{cases} A_1 M^{-2} & \text{for } |x| \geqq \delta \\ A_2 & \text{for } |x| < \delta \end{cases}$$

$$(2.2) \qquad |d_n| \leqq A|n|^{-3}$$

where A, A_1 and A_2 are absolute constants.

Proof. (2.2) is easily proved from the definition of $h(x)$. To prove (2.1) if we put $|x| \geqq \delta$, then by (1.1) and (2.2)

$$|S_M(x)| \leqq |h(x) - S_M(x)| + |h(x)| = |h(x) - S_M(x)|$$
$$\leqq \sum_{|n| \geqq M} A|n|^{-3} \leqq A_1 M^{-2}.$$

On the other hand, if we put $|x| < \delta$, then by the above inequality,

$$|S_M(x)| \leqq |h(x) - S_M(x)| + |h(x)| \leqq A_1 M^{-2} + 1 \leqq A_2.$$

We now define new sequences $\{n_k'\}$, $\{N(\nu)\}$ and $\{M(\nu)\}$, i. e. for $k = 0, \pm 1, \pm 2, \cdots$

$$(2.3) \qquad n_k' = \frac{1}{2}(n_k + n_{k+1})$$

$$(2.4) \qquad N(\nu) = \min(\nu - n_{k-1}, n_{k+1} - \nu)$$

for $n_{k-1}' \leqq \nu \leqq n_k'$ and for $\nu = 0, \pm 1, \pm 2, \cdots$

$$(2.5) \qquad M(\nu) = N(\nu) - 1$$

Lemma 2. *If $\{n_k\}$ satisfies* (1.5), *then*

$$(2.6) \qquad \sum_k \frac{1}{N(n_k)} < \infty$$

Proof. We have for $k=1, 2, \cdots$

$$\frac{N(n_k)}{n_k} = \frac{\min(n_k - n_{k-1}, n_{k+1} - n_k)}{n_k} \leqq \frac{n_{k+1} - n_k}{n_k} < C$$

It follows that

$$\infty > \sum_k \frac{n_k}{N(n_k)^2} > \sum_{k=1}^{\infty} \frac{n_k}{N(n_k)} \cdot \frac{1}{N(n_k)} \geqq \sum_{k=1}^{\infty} \frac{1}{C} \cdot \frac{1}{N(n_k)}$$

This completes the proof.

Since we have from (2.5), (2.4) and (2.3)

$$(2.7)\qquad \begin{aligned} \nu + M(\nu) &< \nu + N(\nu) \leqq \nu + (n_{k+1} - \nu) = n_{k+1}, \\ \nu - M(\nu) &> \nu - N(\nu) \geqq \nu - (\nu - n_{k-1}) = n_{k-1}, \end{aligned}$$

provided that

$$(2.8)\qquad n'_{k-1} \leqq \nu \leqq n'_k,$$

we obtain from (2.7), (2.8) and (1.3)*)

$$c_{n(k)} d_{\nu - n(k)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, S_{M(\nu)}(x) e^{-i\nu x} dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x)\, h(x)\, e^{-i\nu x} dx + \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x) \{S_{M(\nu)}(x) - h(x)\} e^{-i\nu x} dx$$

$$+ \frac{1}{2\pi} \int_{-\pi}^{\pi} \{f(x) - g(x)\} S_{M(\nu)}(x) e^{-i\nu x} dx \equiv I_1 + I_2 + I_3,$$

where $S_{M(\nu)}(x)$ is the $M(\nu)$-th partial sum of the Fourier series of $h(x)$. From the hypotheses of Theorem and Lemma 1, it is obvious that,

$$|I_3| \leqq \frac{1}{2\pi} \int_{|x| \geqq \delta} \{|f(x)| + |g(x)|\} |S_{M(\nu)}(x)| dx \leqq O(M(\nu)^{-2})$$

$$|I_2| \leqq \frac{1}{2\pi} \int_{-\pi}^{\pi} |g(x)| |h(x) - S_{M(\nu)}(x)| dx \leqq O(M(\nu)^{-2})$$

We shall now consider the series $\sum M(\nu)^{-2}$. From (2.5), (2.3) and Lemma 2, it follows that

$$\sum_{\nu=0}^{\infty} \frac{1}{M(\nu)^2} = \sum_{k=1}^{\infty} \sum_{\nu=n'_{k-1}}^{n'_k - 1} \frac{1}{M(\nu)^2} + \sum_{\nu=0}^{n'_0 - 1} \frac{1}{M(\nu)^2}$$

$$\leqq \sum_{k=1}^{\infty} \left(\sum_{j=n'_{k-1} - n_{k-1}}^{\infty} \frac{1}{(j-1)^2} + \sum_{j=n_{k+1} - n'_k}^{\infty} \frac{1}{(j-1)^2} \right) + \sum_{j=n_1 - n'_0}^{\infty} \frac{1}{(j-1)^2}$$

$$\leqq O(1) \left(\sum_{k=1}^{\infty} \frac{2}{n_k - n_{k-1}} + \sum_{k=1}^{\infty} \frac{2}{n_{k+1} - n_k} + \frac{2}{n_1 - n_0} \right)$$

$$\leqq O(1) \sum_{k=1}^{\infty} \frac{1}{n_k - n_{k-1}} \leqq O(1) \sum_{k=0}^{\infty} \frac{1}{N(n_k)} < \infty,$$

so that we have under the hypotheses of Theorem, the absolute convergence of

$$\sum_{\nu=0}^{\infty} |(I_2 + I_3) e^{i\nu x}| < \infty.$$

Similarly, we have

$$\sum_{\nu=-\infty}^{-1} |(I_2 + I_3) e^{i\nu x}| < \infty.$$

Hence we have that

*) Hereafter $n(k)$ means n_k $(k = 0, \pm 1, \pm 2, \cdots)$.

$$\sum_{-\infty}^{\infty} (I_2 + I_3)\, e^{i\nu x}$$

converges absolutely and as a matter of course it is $|C, 1|$ - summable at every point x.

Applying Theorem B, the Fourier series $\sum I_1 e^{i\nu x}$ of $g(x)\, h(x)$ is $|C, 1|$ - summable at every point x, and we obtain the following

Lemma 3. *Under the hypotheses of Theorem, the trigonometric series*

$$(2.9) \qquad \sum_{k=-\infty}^{\infty} \sum_{\nu=n'_{k-1}}^{n'_k} c_{n(k)}\, d_{\nu-n(k)}\, e^{i\nu x}$$

is $|C, 1|$ - *summable at every point* x.

Lemma 4. *Let* $\{n_{2k}\,; k = 0, \pm 1, \pm 2, \cdots\}$ *satisfy* (1.5) (a). *If*

$$(2.10) \qquad n_{2k+1} = (n_{2k} + n_{2k+2})\,/\,2,$$

then we have

$$(2.11) \qquad \Delta \equiv \Big(\sum_{m=n_{2k}}^{n_{2k+2}} \frac{1}{m^2}\Big)\Big/\Big(\sum_{m=n'_{2k}}^{n'_{2k+1}} \frac{1}{m^2}\Big) = O(1).$$

Proof.

$$\Delta = 1 + \Big(\sum_{m=n_{2k}}^{n'_{2k}} \frac{1}{m^2} + \sum_{m=n'_{2k+1}}^{n_{2k+2}} \frac{1}{m^2}\Big)\Big/\Big(\sum_{m=n'_{2k}}^{n'_{2k+1}} \frac{1}{m^2}\Big)$$

$$\leqq 1 + \Big\{\Big(\frac{1}{n_{2k}}\Big)^2 + \Big(\frac{1}{n'_{2k+1}}\Big)^2\Big\}\Big/ 2\Big(\frac{1}{n'_{2k+1}}\Big)^2$$

$$= 1 + \Big(\frac{1}{2} + \frac{1}{2}\Big(\frac{n'_{2k+1}}{n_{2k}}\Big)^2\Big) = \frac{3}{2} + \frac{1}{2}\Big(1 + \frac{3}{4}\,\frac{n_{2k+2} - n_{2k}}{n_k}\Big)^2$$

$$\leqq \frac{3}{2} + \frac{1}{2}\,(1 + \frac{3}{4}\,C)^2 < \infty.$$

This completes the proof.

3. To prove Theorem, we may suppose without any loss of generality that $\{n_{2k};\ k = 0, \pm 1, \pm 2, \cdots\}$ satisfies (1.3) — (1.5) and $\{n_{2k+1}\,;\ k = 0, \pm 1, \pm 2, \cdots\}$ satisfies (2.10) and

$$(3.1) \qquad c_{n(2k+1)} = 0 \qquad\qquad k = 0, \pm 1, \pm 2, \cdots$$

For any positive integer m, we put

$$(3.2) \qquad A(m) = \sum_{|\nu| \leqq m} |\nu|\, p_\nu\, e^{i\nu x},$$

where

$$(3.3) \qquad p_\nu = c_{n(k)}\, d_{\nu-n(k)} \qquad \text{for} \qquad n'_{k-1} \leqq \nu \leqq n'_k,$$

$k=0,\ \pm 1,\ \pm 2,\ \cdots$ so that by (3.1) we have

$$(3.4)\qquad A(m)=A(n'_{2k})\qquad \text{for}\qquad n'_{2k}\leqq m\leqq n'_{2k+1}$$

From Lemma 3 and the definition of $|C,1|$ - summability, we have

$$(3.5)\qquad \infty>\sum_{m=1}^{\infty}\frac{1}{m^2}\,|A(m)|\geqq\sum_{k=1}^{\infty}\sum_{n'_{2k}\leqq m\leqq n'_{2k+1}}\frac{1}{m^2}\,|A(n'_{2k})|$$

we must estimate $A(n'_{2k})$ more precisely.

$$(3.6)\qquad A(n'_{2k})=\sum_{j=1}^{k}\Big(\sum_{\nu=n'_{2j-1}}^{n'_{2j}}\nu\,p_\nu e^{i\nu x}+\sum_{\nu=n'_{-2j-1}}^{n'_{-2j}}(-\nu)\,p_\nu e^{i\nu x}\Big)$$

$$+\sum_{\nu=n'_{-1}}^{n'_0}|\nu|\,p_\nu e^{i\nu x}\equiv P+Q+R.$$

A little more precise formula of P is as follows.

$$P=\sum_{j=1}^{k}\sum_{\mu=n'_{2j-1}-n_{2j}}^{n'_{2j}-n_{2j}}(\mu+n_{2j})\,c_{n(2j)}\,e^{i(\mu+n_{2j})x}d_\mu$$

$$=\Big(\sum_{j=1}^{k}n_{2j}\,c_{n(2j)}\,e^{in_{2j}x}\Big)\Big\{h(x)-\Big(\sum_{\mu>n'_{2j}-n_{2j}}+\sum_{\mu<-(n_{2j}-n'_{2j-1})}\Big)d_\mu\,e^{i\mu x}\Big\}$$

$$+\sum_{j=1}^{k}c_{n(2j)}\,e^{in_{2j}x}\sum_{\mu=-(n_{2j}-n'_{2j-1})}^{n'_{2j}-n_{2j}}\mu d_\mu\,e^{i\mu x}\equiv P_1+P_2+P_3.$$

It follows from (1.5) (c) that, noticing $\{c_n\}$ is uniformly bounded,

$$(3.7)\qquad |P_2|\leqq\sum_{j=1}^{k}|n_{2j}\,c_{n(2j)}|\Big(\frac{1}{(n'_{2j}-n_{2j})^2}+\frac{1}{(n_{2j}-n'_{2j-1})^2}\Big)$$

$$\leqq A\sum_{j=1}^{\infty}\frac{n_{2j}}{N(n_{2j})^2}<\infty$$

$$(3.8)\qquad |P_3|\leqq\sum_{j=1}^{k}|c_{n(2j)}|\sum_{\mu\neq 0}\frac{1}{\mu^2}\leqq 2Ak$$

Similarly if we write Q in the following formula

$$Q=\Big(\sum_{j=1}^{k}|n_{-2j}|\,c_{n(-2j)}e^{in_{-2j}x}\Big)\Big\{h(x)-\Big(\sum_{\mu<n'_{-2j}-n_{-2j}}+\sum_{\mu<-(n_{-2j}-n'_{2j-1})}\Big)d_\mu\,e^{i\mu x}$$

$$+\sum_{j=1}^{k}c_{n(-2j)}e^{in_{-2j}x}\sum_{\mu=-(n_{-2j}-n'_{-2j-1})}^{n'_{-2j}-n_{-2j}}(-\mu)\,d_\mu\,e^{i\mu x}\equiv Q_1+Q_2+Q_3,$$

then we have

$$(3.7')\qquad |Q_2|\leqq A\sum_{j=1}^{\infty}\frac{n_{2j}}{N(n_{2j})^2}<\infty$$

$$(3.8') \qquad | Q_3 | \leqq 2Ak$$

Concequently, we obtain by (3.6), (3.7), (3.7′), (3.8) and (3.8′).

$$(3.9) \qquad | P+Q+R | \geqq | P+Q | - | R | \geqq h(x) \left| \sum_{|n_l| \leqq n_{2k}} | n_l | \, c_{n(l)} e^{i n_l x} \right| - Ak,$$

and thus from (3.5), (3.6), (3.9) and (1.5) (b)

$$(3.10) \qquad \infty > h(x) \sum_{k=1}^{\infty} \sum_{n'_{2k} \leqq m \leqq n'_{2k+1}} \frac{1}{m^2} \left| \sum_{|n_l| \leqq m} | n_l | \, c_{n(l)} e^{i n_l x} \right| - A$$

Since $h(x) > 0$ in the interval $(-\delta, \delta)$, (3.10) leads

$$(3.11) \qquad \sum_{k=1}^{\infty} \sum_{n'_{2k} \leqq m \leqq n'_{2k+1}} \frac{1}{m^2} \left| \sum_{|n_l| \leqq m} | n_l | \, c_{n(l)} e^{i n_l x} \right| < \infty,$$

Now applying Lemma 4 and (3.11),

$$\sum_{m=1}^{\infty} \frac{1}{m^2} \left| \sum_{|n_l| \leqq m} | n_l | \, c_{n(l)} e^{i n_l x} \right| = \sum_{k=1}^{\infty} \sum_{m=n_{2k}}^{n_{2k+2}-1} \frac{1}{m^2} \left| \sum_{|n_l| \leqq m} | n_l | \, c_{n(l)} e^{i n_l x} \right|$$

$$= \sum_{k=1}^{\infty} \sum_{m=n_{2k}}^{n_{2k+2}-1} \frac{1}{m^2} \left| \sum_{|n_l| \leqq n_{2k}} | n_l | \, c_{n(l)} e^{i n_l x} \right|$$

$$= \sum_{k=1}^{\infty} \sum_{m=n'_{2k}}^{n'_{2k+1}} \frac{1}{m^2} \left| \sum_{|n_l| \leqq n_{2k}} | n_l | \, c_{n(l)} e^{i n_l x} \right| \cdot \Delta < \infty$$

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