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# Absolute convergence of Fourier series with a gap condition

by

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**1**. Let the $L$-integrable function $f(x)$, with period $2\pi$, have the Fourier series

$$\frac{1}{2}a_0 + \sum_{p=1}^{\infty}(a_p \cos px + b_p \sin px), \tag{1.1}$$

where

$$a_p = b_p = 0\,, \text{ for } p \neq n_k, \qquad p = 0, 1, 2, \ldots$$

$$\lim_{k\to\infty} \frac{N_k}{\log n_k} = \infty\,, \quad N_k = \min(n_{k+1} - n_k, n_k - n_{k-1}), \tag{1.2}$$

Concering the order of the Fourier coefficients of $f(x)$ and the absolute convergence of the Fourier series of $f(x)$, N. E. Noble[ 1 ], P. B. Kennedy [2,3], M. Tomić [4] and S. M. Mazhar [5] proved very interestiog results. One of the Noble's assertions is as follows:

*Suppose* (1.1) *and* (1.2), *and* $f(x)$ *satisfies a Lipschitz condition of order* $\alpha$, $0 < \alpha < 1$, *in some interval* $|x - x_0| \leqq \delta$, *then for* $\beta < \alpha$

$$\sum_{p=1}^{\infty} p^{\beta-\frac{1}{2}} (\,|a_p| + b_p|\,) < \infty$$

In the present paper we have tried to obtain a kind of analogous modification of the following well known Wiener's theorem.

*Let* $F(x)$ *be Lebesgue integrable in the interval* $(-\pi, \pi)$ *with period* $2\pi$. *If at every point* $y$ *on the closed interval* $[-\pi, \pi]$ *there exist a function* $G_y(x)$ *and a* $\delta > 0$ *such that* (i) $G_y(x) = F(x)$ *for* $|x - y| < \delta$, *and* (ii) *the Fourier series of* $G_y(x)$ *is absolutely convergent, then the Fourier series of* $F(x)$ *is absolutely convergent.*

We shall show that the following theorem will be established.

*Theorem*. Let $f(x)$ satisfy (1.1) and (1.2). If there exist an interval $|x - x_0| < \delta$ and a function $g(x)$ such that

(i) $\qquad g(x) = f(x) \qquad$ for $|x - x_0| < \delta$

(ii) $\qquad g(x) \sim \frac{1}{2}\alpha_0 + \sum(\alpha_p \cos px + \beta_p \sin px)$

(iii) $$\sum_{p=1}^{\infty}(|\alpha_p|+|\beta_p|)\omega(p)<\infty,$$

where $\omega(x)$ is a function defined on $1\leqq x<\infty$, and satisfy for any $y\geqq 1$, and absolute constants $c$, $c_1$ and $c_2$

(1.3) $$\frac{\max(\omega(x)\,;y\leqq x\leqq 8y)}{\min(\omega(x)\,;y\leqq x\leqq 8y)}\leqq c_1<\infty\ , \qquad 0<\omega(x)\leqq c_2x^c$$

then $$\sum_{p=1}^{\infty}(|a_p|+|b_p|)\omega(p)<\infty$$

As a special case, if we take $\omega(p)=1$, for $p=1,2,\ldots$, then we obtain the above mentioned Noble's type theorem of Wiener's one. Moreover if we take, on $[-\pi,\pi)$

$g(x)\in \mathrm{Lip}\alpha$ , $\frac{1}{2}<\alpha\leqq 1$ and $\omega(p)=1$, $p=1,2,\ldots$

$g(x)\in \mathrm{Lip}\alpha$ , $0<\alpha\leqq 1$ and $g(x)\in \mathrm{BV}$ and $\omega(p)=1$, $p=1,2,\ldots$

$g(x)\in \mathrm{Lip}\alpha$ , $0<\alpha\leqq 1$ and $\omega(p)=p^{\beta-\frac{1}{2}}$ , $\beta<\alpha$ , $p=1,2,\ldots$

$g(x)\in \mathrm{Lip}\alpha$ , $0<\alpha\leqq 1$ and $g(x)\in\mathrm{BV}$, $\omega(p)=p^{\beta/2}$, $\beta<\alpha$, $p=1,2,\ldots$

then from our theorem, we obtain Theorem 3, Theorem 5 and Theorem 4 (ii) of Noble's paper, respectively.

**2.** We suppose, without any loss of generality, that for each $m=0,1,2,\ldots$ there exists at least one $n_k$ such that

(2.1) $$2^m\leqq n_k<2^{m+2}$$

If otherwise, we make $2^{m+1}$ a term of the sequence $\{n_k\}$. Moreover we may assume that $x_0=0$.

Putting for each $k=1,2,\ldots$

(2.2) $$f_k(x)=f\left(x+\frac{\pi}{2n_k}\right)-f\left(x-\frac{\pi}{2n_k}\right),$$
$$g_k(x)=g\left(x+\frac{\pi}{2n_k}\right)-g\left(x-\frac{\pi}{2n_k}\right),$$

then there exists an integer $k_0$ $(=k_0(\delta))$ such that for $k\geqq k_0$ and $|x|<\frac{\beta}{2}$,

(2.3) $$f_k(x)=g_k(x),$$

and the Fourier series of $f_k(x)$ and $g_k(x)$ are, respectively,

(2.4) $$f_k(x)\sim\sum_{p=0}^{\infty}2\left(\sin\frac{p\pi}{2n_k}\right)(b_p\cos px-a_p\sin px),$$
$$g_k(x)\sim\sum_{p=0}^{\infty}2\left(\sin\frac{p\pi}{2n_k}\right)(\beta_p\cos px-\alpha_p\sin px).$$

Now if we put for $i=1,2,\ldots$

$$M_i=\min(N_i,\,n_i)-1,$$

then the sequence of integral numbers $\{M_i\}$ satisfies

$$(2.5) \qquad M_i < N_i, \quad M_i < n_i, \quad \lim_{i\to\infty} \frac{N_i}{\log n_i} = \infty$$

We consider the sequence of trigonometric polynomials $\{T_M(x)\}$, which was defined by Noble [1] and has the following properties,

$$(2.6) \qquad \begin{aligned} &\text{(i)} && T_M(x) = \sum_{p=0}^{M} d_p \cos px, \quad d_0 = 1, \\ &\text{(ii)} && |T_M(x)| \leqq A_1 \frac{1}{\delta} \quad \text{for } |x| < \frac{\delta}{2}, \\ &\text{(iii)} && |T_M(x)| \leqq A_2 \exp(-A_3 \delta M) \quad \text{for } |x| \geqq \frac{\delta}{2}, \end{aligned}$$

where $A_1$, $A_2$ and $A_3$ are some absolute constants.

By (2.6)(i), the frequency of each terms of $T_{M_i}(x) \cos n_i x$ lies in the interval $(n_i - M_i,\ n_i + M_i)$, and

$$(2.7) \qquad \begin{aligned} n_i - M_i &> n_i - N_i \geqq n_i - (n_i - n_{i-1}) = n_{i-1} \\ n_i + M_i &< n_i + N_i \leqq n_i + (n_{i+1} - n_i) = n_{i+1} \end{aligned}$$

By (1.1), (1.2), (2.4) and (2.7), we have

$$2 b_{n_i} \sin \frac{\pi n_i}{2 n_k} = \frac{1}{\pi} \int_{-\pi}^{\pi} f_k(x) \cos n_i x \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} f_k(x) T_{M_i}(x) \cos n_i x \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} g_k(x) T_{M_i}(x) \cos n_i x \, dx + \frac{1}{\pi} \int_{-\pi}^{\pi} (f_k(x) - g_k(x)) T_{M_i}(x) \cos n_i x \, dx$$

$$= p_i + q_i.$$

By (2.3), (2.6) and (2.2)

$$|q_i| \leqq \frac{1}{\pi} \int_{|x| \geqq \frac{\delta}{2}} (|f_k(x)| + |g_k(x)|) A_2 \exp(-A_3 \delta M_i) \, dx$$

$$\leqq A_2 \exp(-A_3 \delta M_i) \int_{-\pi}^{\pi} 2(|f(x)| + |g(x)|) \, dx \leqq A_4 \exp(-A_3 \delta M_i)$$

On the other hand $p_i$, $i = 1, 2, \ldots$, is more complicated. By reason of (2.4), (2.6), (2.5) and (2.7)

$$p_i = \frac{1}{\pi} \sum_{p=0}^{\infty} 2 \left(\sin \frac{\pi p}{2 n_k}\right) \int_{-\pi}^{\pi} (\beta_p \cos px - \alpha_p \sin px) T_{M_i}(x) \cos n_i x \, dx$$

$$= \sum_{p=0}^{\infty} \left(\sin \frac{\pi p}{2 n_k}\right) \Big(\beta_p \cdot \frac{1}{\pi} \int_{-\pi}^{\pi} T_{M_i}(x)\, 2 \cos px \cos n_i x \, dx$$

$$- \alpha_p \cdot \frac{1}{\pi} \int_{-\pi}^{\pi} T_{M_i}(x)\, 2 \sin px \cos n_i x \, dx\Big)$$

$$= \sum_{p=o}^{\infty} (\sin \frac{\pi p}{2 n_k}) \cdot \beta_p \cdot \frac{1}{\pi} \int_{-\pi}^{\pi} T_{M_i}(x) \cos(p-n_i)x \, dx$$

$$= \sum_{o \leqq p-n_i \leqq M_i} (\sin \frac{\pi p}{2 n_k}) \cdot p_p d_{p-n_i} + \sum_{o \leqq n_i - p \leqq M_i} (\sin \frac{\pi p}{2 n_k}) \cdot \beta_p d_{n_i - p}$$

Since by (2.6) we have, $|d_p| \leqq A_5$ for $p=0,1,2,\ldots M_i$, uniformly in $i$,

$$|p_i| \leqq A_5 \sum_{n_{i-1}<p<n_{i+1}} |\beta_p|$$

and for $k \leqq k_0$

$$\left| 2 b_{n_i} \sin \frac{\pi n_i}{2 n_k} \right| \leqq A_4 \exp(-A_3 \delta M_i) + A_5 \sum_{n_{i-1}<p<n_{i+1}} |\beta_p|$$

By the same reason we obtain, for $k \geqq k_0$,

$$\left| 2 a_{n_i} \sin \frac{\pi n_i}{2 n_k} \right| \leqq A_4 \exp(-A_3 \delta M_i) + A_5 \sum_{n_{i-1}<p<n_{i+1}} |\alpha_p|$$

Thus we obtain

$$(2.8) \quad (|a_{n_i}| + b_{n_i}|) \leqq A_6 \exp(-A_3 \delta M_i) + A_5 \sum_{n_{i-1}<p<n_{i+1}} (|\alpha_p| + |\beta_p|)$$
$$\equiv P_i + Q_i,$$

provided that

$$(2.9) \qquad k \geqq k_o, \qquad \frac{1}{2} n_k < n_i \leqq n_k.$$

If we denote the largest $k$ such as $2^{m-1} < n_k \leqq 2^m$ by $k' = k'(m)$ and such as $n_k < \frac{1}{2} \cdot n_{k1}$, by $k'' = k''(m)$, then by (2.8) we have

$$\sum_{i=1}^{\infty} (|a_{n_i}| + |b_{n_i}|) \omega(n_i) = \sum_{m=1}^{\infty} \sum_{2^{m-1} < n_i \leqq 2^m} (|a_{n_i}| + |b_{n_i}|) \omega(n_i) + A_7$$

$$\leqq \sum_{m=1}^{\infty} \sum_{2^{m-1} < n_i \leqq 2^m} P_i \omega(n_i) + \sum_{m=1}^{\infty} \sum_{2^{m-1} < n_i \leqq 2^m} Q_i \omega(n_i) + A_7$$

$$= A_6 \sum_{m=1}^{\infty} \sum_{\frac{1}{2} n_{k'} < n_i < n_{k'}} \omega(n_i) P_i + \sum_{m=1}^{\infty} \sum_{\frac{1}{2} n_{k'} < n_i < n_{k'}} Q_i \omega(n_i) + A_7$$
$$\equiv P + Q + A_7$$

It is easy to see, by (2.5) and (1.3)

$$(2.10) \qquad P = 2 A_6 \sum_{i=1}^{\infty} \omega(n_i) \exp(-A_3 \delta M_i) < \sum_{i=1}^{\infty} \frac{1}{n_i^2} \leqq \sum_{i=1}^{\infty} \frac{1}{i^2} < \infty$$

On the other hand by (2.8)

$$Q = \sum_{m=1}^{\infty} \sum_{\frac{1}{2} n_{k'} < n_i \leqq n_{k'}} \left\{ \omega(n_i) \sum_{n_{i-1} < p \leqq n_i} (|\alpha_p| + |\beta_p|) \right.$$
$$\left. + \omega(n_i) \sum_{n_i < p < n_{i+1}} (|\alpha_p| + |\beta_p|) \right\}$$

$$= \sum_{m=1}^{\infty} \Big\{ \sum_{\frac{1}{2}n_{k'}<n_i<n_{k'}} \sum_{n_i<p\leqq n_{i+1}} (|\alpha_p|+|p_p|)(\omega(n_i)+\omega(n_{i+1}))$$

$$+ \sum_{n_{k''}<p\leqq n_{k''+1}} (|\alpha_p|+|\beta_p|)\omega(n_{k''+1}) + \sum_{n_{k'}<p\leqq n_{k'+1}} (|\alpha_p|+|\beta_p|)\omega(n_{k'}) \Big\}$$

$$= 2 \sum_{i=1}^{\infty} \sum_{n_i<p\leqq n_{i+1}} (|\alpha_p|+|\beta_p|)(\omega(n_i)+\omega(n_{i+1}))$$

$$= 2 \sum_{i=1}^{\infty} \sum_{n_i<p\leqq n_{i+1}} \frac{\omega(n_i)+\omega(n_{i+1})}{\min(\omega(p);n_i<p\leqq n_{i+1})} (|\alpha_p|+|\beta_p|)\omega(p)$$

$$= A_8 \sum_{i=1}^{\infty} (|\alpha_p|+|\beta_p|)\omega(p)$$

This last formular follows from (1.3) and (2.1). From this and (2.10) we obtain

$$\sum_p (|a_p|+|b_p|)\omega(p) \leqq A_9 \sum_p (|\alpha_p|+|\beta_p|\omega(p)<\infty$$

### References


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