Sci. Rep. Kanazawa Univ.,
Vol. 9, No. pp. 155–164
Dec. 1964

On Some Maximal Theorems for the Gap Series

Noboru MATSUYAMA*

(Received 25 Sept. 1964)

1. Introduction. In this note let $f(x)$ be an integrable function with period 1, and satisfy

$$(1.1) \qquad \int_0^1 f(x)\,dx=0, \qquad \int_0^1 f(x)^2dx=1,$$

and let $S_n(x)$ be the n-th order partial sum of the Fourier series of $f(x)$.

Our object of this note is to prove the convergence and the (C)-summability of gap series

$$(1.2) \qquad \sum_{k=0}^{\infty} c_k f(n_k x),$$

where $\{n_k\}$ being a sequence of positive integers, satisfies

$$(1.3) \qquad n_{k+1}/n_k \geqq c > 1 \qquad (k=0,1,2,\ldots)$$

Actually we prove the following theorems:

Theorem 1. If, for $\alpha>1$, $f(x)$ satisfies

$$(1.4) \qquad \left(\int_0^1 \left[f(x)-S_n(x)\right]^2 dx\right)^{1/2}=O\left((\log n)^{-\alpha}\right),$$

as $n\to\infty$, then for every non-negative integral number q, there exist some constants $A_{q,c}$ and B depending**) only on q and c such that,

$$(1.5) \qquad \int_0^1 \sup_N \left[\sum_{k=0}^{N} c_k f(n_k x)\right]^2 dx \leqq A_{q,c} \sum_{k=0}^{\infty} c_k^2 \left(\log_{q+1}(k+B)\right)^2$$

Theorem 2. If, for $\alpha>1$, $f(x)$ satisfies (1.4), then for any $\beta>0$,

$$\int_0^1 \sup_N \left[\sigma_N^{(\beta)}(x)\right]^2 dx \leqq A \sum_{k=0}^{\infty} c_k^2,$$

where $\sigma_N^{(\beta)}(x)$ denotes the (C,β)-mean of the series (1.2).

Theorem 1 is analogous to Kantorovitch's one, being a maximal theorem for the orthogonal series (see [1] and [2]), and it is also a generalization of the well known

* *Department of Mathematics, Faculty of Science, Kanazawa University*

** Hereafter we suppose that A, A_q,……are constants which may be different instances, and $\log_q$ means the iterated logarithm of q times.

Kac-Salem-Zygmund theorem [3]. Now it is interesting to compare Theorem 1 and Theorem 2 for the corresponding theorems of the orthogonal series, especially Kaczmarz-Menchoff theorem [5] and Rademacher-Khintchine-Kolmogoroff theorem [6].

2. Lemmas. In this section we prove several lemmas, which need for the proof of our theorems.

Lemma 1. If $\{n_k\}$ satisfies (1.3), then

$$\int_0^1 \sup_N (S_{n_N}(x))^2 dx \leqq A_c \int_0^1 f(x)^2 dx$$

This is well known (see [4]).

Lemma 2. Let $M(k)$ and $N(k)$ be two non-decreasing sequences and $1 \leqq M(k) < N(k)$. If, for $\alpha > 1$, $f(x)$ satisfies (1.4), then we have for any posititive integers p and q $(p<q)$,

$$\text{(i)} \quad \int_0^1 \Big(\sum_{k=p}^{q} c_k \big[S_{N(k)}(n_k x) - S_{M(k)}(n_k x) \big] \Big)^2 dx \leqq A \sum_{k=p}^{q} c_k^2 (\log M(k))^{-\alpha},$$

$$\text{(ii)} \quad \int_0^1 \Big(\sum_{k=p}^{q} c_k S_{N(k)}(n_k x) \Big)^2 dx \leqq A \sum_{k=p}^{q} c_k^2$$

Proof. By (1.4), it is easy to verify that, for $j<k$,

$$(2.1) \quad \int_0^1 \big[S_{N(j)}(n_j x) - S_{M(j)}(n_j x) \big] \big[S_{N(k)}(n_k x) - S_{M(k)}(n_k x) \big] dx$$
$$\leqq A (k-j)^{-\alpha} (\log M(k))^{-\alpha}.$$

Hence from (2.1) and (1.4), we have the following estimations,

$$\int_0^1 \Big(\sum_{k=p}^{q} c_k \big[S_{N(k)}(n_k x) - S_{M(k)}(n_k x) \big] \Big)^2 dx = \sum_{k=p}^{q} c_k^2 \int_0^1 \big[S_{N(k)}(n_k x) - S_{M(k)}(n_k x) \big]^2 dx$$
$$+ 2 \sum_{k>j} c_k c_j \int_0^1 \big[S_{N(j)}(n_j x) - S_{M(j)}(n_j x) \big] \big[S_{N(k)}(n_k x) - S_{M(k)}(n_k x) \big] dx$$
$$\leqq A \sum_{k=p}^{q} c_k^2 (\log M(k))^{-2\alpha} + 2A \sum_{k=p+1}^{q} \sum_{j=p}^{k-1} c_k c_j (k-j)^{-\alpha} (\log M(k))^{-\alpha}$$
$$= A \sum_{k=p}^{q} c_k^2 (\log M(k))^{-2\alpha} + 2A \sum_{k=p+1}^{q} \sum_{r=1}^{k-p} r^{-\alpha} c_k c_{k-r} (\log M(k))^{-\alpha}$$
$$= A \sum_{k=p}^{q} c_k^2 (\log M(k))^{-2\alpha} + 2A \sum_{r=1}^{q-p} r^{-\alpha} \sum_{k=p+r}^{q} c_k (\log M(k))^{-\alpha/2} c_{k-r} (\log M(k))^{-\alpha/2}$$
$$\leqq A \sum_{k=p}^{q} c_k^2 (\log M(k))^{-2\alpha} + 2A \sum_{k=p}^{q} c_k^2 (\log M(k))^{-\alpha} \sum_{r=1}^{q-p} r^{-\alpha}$$
$$\leqq A \sum_{k=p}^{q} c_k^2 (\log M(k))^{-\alpha}.$$

This is the proof of (i), and similarly we can prove (ii).

Let $U_N^{(\varepsilon)}(x)$ be the (C,ε)-mean of $\sum c_k S_{\tau(k)}(n_k x)$, that is,

$$(2.2)\qquad U_N^{(\varepsilon)}(x)=\frac{1}{A_N^{(\varepsilon)}}\sum_{k=0}^{N}A_{N-k}^{(\varepsilon)}c_k S_{\tau(k)}(n_k x),$$

where

$$(2.3)\qquad \tau(k)=[\exp\sqrt{k+1}],$$

then we have the following two lemmas;

Lemma 3. If ε satisfies

$$(2.4)\qquad \frac{1}{2}<\varepsilon\leqq 1,$$

then

$$(2.5)\qquad \sum_{n=0}^{\infty}\int_0^1 \sup_{2^n<m<2^{n+1}}\left[U_m^{(\varepsilon)}(x)-U_{2^n}^{(\varepsilon)}(x)\right]^2 dx\leqq A\sum_{k=0}^{\infty}c_k^2$$

proof. Firstly we suppose $\varepsilon\neq 1$, then from the definition (2.2), we have

$$(2.6)\qquad U_n^{(\varepsilon)}(x)-U_{n-1}^{(\varepsilon)}(x)=\frac{\varepsilon}{nA_n^{(\varepsilon)}}\sum_{k=0}^{n-1}A_{n-k}^{(\varepsilon)}\frac{1}{n-k}kc_k S_{\tau(k)}(n_k x)$$

$$+\frac{1}{A_n^{(\varepsilon)}}c_k S_{\tau(n)}(n_k x)$$

By (2.4), and (2.6) and Lemma 2 (ii),

$$\sum_{n=0}^{\infty}\int_0^1 \sup_{2^n<m<2^{n+1}}\left[U_m^{(\varepsilon)}(x)-U_{2^n}^{(\varepsilon)}(x)\right]^2 dx$$

$$\leqq A\sum_{n=0}^{\infty}\int_0^1\sum_{k=2^n+1}^{2^{n+1}}k\left[U_k^{(\varepsilon)}(x)-U_{k-1}^{(\varepsilon)}(x)\right]^2 dx=A\sum_{n=2}^{\infty}\int_0^1 n\left[U_n^{(\varepsilon)}(x)-U_{n-1}^{(\varepsilon)}(x)\right]^2 dx$$

$$\leqq A\sum_{n=2}^{\infty}n\sum_{k=0}^{n-1}\left(\frac{\varepsilon}{nA_n^{(\varepsilon)}}\right)^2\left(\frac{A_{n-k}^{(\varepsilon)}kc_k}{n-k}\right)^2+A\sum_{n=2}^{\infty}nc_n^2\frac{1}{\left(A_n^{(\varepsilon)}\right)^2}$$

$$\leqq A\sum_{k=0}^{\infty}k^2c_k^2\sum_{n=k+1}^{\infty}\frac{(n-k)^{2\varepsilon-2}}{n^{1+2\varepsilon}}+A\sum_{n=0}^{\infty}c_n^2=A\sum_{k=0}^{\infty}k^2c_k^2\left(\sum_{n=k+1}^{2k}+\sum_{n=2k+1}^{\infty}\right)\frac{(n-k)^{2\varepsilon-2}}{n^{1+2\varepsilon}}$$

$$\leqq A\sum_{k=0}^{\infty}k^2c_k^2\left(\frac{1}{k^{1+2\varepsilon}}\sum_{i=1}^{k}\frac{1}{i^{2-2\varepsilon}}+\frac{1}{(k+1)^{2-2\varepsilon}}\sum_{n=2k+1}^{\infty}\frac{1}{n^{1+2\varepsilon}}\right)$$

$$\leqq A\sum_{k=0}^{\infty}k^2c_k^2\left(\frac{1}{k^{1+2\varepsilon}}\frac{1}{k^{1-2\varepsilon}}+\frac{1}{k^{2-2\varepsilon}}\frac{1}{k^{2\varepsilon}}\right)\leqq A\sum_{k=0}^{\infty}c_k^2$$

This is (2.5).

Secondary if $\varepsilon=1$, we obtain (2.5) by the same way.

Lemma 4. If $\beta>1$, then

$$(2.7)\qquad \int_0^1\sup_N\left[U_{2^N}^{(\beta)}(x)\right]^2 dx\leqq A\int_0^1\sup_N\left[U_N(x)\right]^2 dx,$$

holds, where $U_N(x)$ means $U_N^{(1)}(x)$.

Proof. By the definition (2.2)

$$\left[U_{2^N}^{(\beta)}(x)\right]^2=\frac{1}{\left(A_{2^N}^{(\beta)}\right)^2}\left[\sum_{k=0}^{2^N} A_{2^N-k}^{(\beta-2)}(k+1)U_k(x)\right]^2$$

$$\leqq\left(\sup_N\left[U_N(x)\right]^2\right)\frac{1}{\left(A_N^{(\beta)}\right)^2}\left(\sum_{k=0}^{2^N}(k+1)A_{2^N-k}^{(\beta-2)}\right)^2$$

$$\leqq\left(\sup_N\left[U_N(x)\right]^2\right)\frac{A}{2^{N2\beta}}(2^N+1)^2(A_{2^N}^{(\beta-1)})^2\leqq A\sup_N\left[U_N(x)\right]^2$$

Thus we obtain (2.7).

3. Proof of Theorem 1. In order to prove Theorem 1, we may suppose, for some positive integer q,

$$\text{(3.1)}\qquad \sum_{k=0}^{\infty} c_k^2\left(\log_{q+1}(k+B)\right)^2<+\infty$$

where $B=B_{q,c}$ is a constant determined by

$$\text{(3.2)}\qquad \min\left((\log_q B)^{2/\alpha},\ \frac{2}{\log c}(\log_q B)^{2/\alpha}\right)\geqq 1.$$

Now let us put, for $p=1,2,3,\ldots,q$,

$$\text{(3.3)}\qquad \tau_0(n)=[\exp\lambda_0(n)],\qquad \lambda_0(n)=\sqrt{n+1},$$

$$\tau_p(n)=[\exp\lambda_p(n)],\qquad \lambda_p(n)=(\log_p(n+B))^{2/\alpha},$$

then by (3.2), each $\lambda_0(n)$, $\lambda_1(n),\ldots,\lambda_q(n)$ and consequently each $\tau_0(n)$, $\tau_1(n),\ldots$, $\tau_q(n)$ is respectively non-decreasing for $n=0,1,2,\ldots$

Moreover, we put for $l=0,1,\ldots,q-1$

$$Y_{-1}(x)=\sup_N\left(\sum_{k=0}^{N} c_k\left[f(n_kx)-S_{\tau_0(k)}(n_kx)\right]\right)^2$$

$$\text{(3.1)}\qquad Y_l(x)=\sup_N\left(\sum_{k=0}^{N} c_k\left[S_{\tau_l(k)}(n_kx)-S_{\tau_{l+1}(k)}(n_kx)\right]\right)^2$$

$$Y_q(x)=\sup_N\left(\sum_{k=0}^{N} c_kS_{\tau_q(k)}(n_kx)\right)^2,$$

then we easily have

$$\text{(3.5)}\qquad \int_0^1\sup_N\left(\sum_{k=0}^{N} c_kf(n_kx)\right)^2dx\leqq A_q\sum_{l=-1}^{q}\int_0^1 Y_l(x)dx,$$

and from this, we obtain Theorem 1, provided that for $l=-1,0,1,\ldots,q$

$$\text{(3.6)}\qquad \int_0^1 Y_l(x)\,dx\leqq A_c\sum_{k=0}^{\infty} c_k^2\left(\log_{q+1}(k+B)\right)^2$$

Firstly we estimate $Y_{-1}(x)$.

$$\int_0^1 Y_{-1}(x)\,dx=\int_0^1 \sup_N \sum_{k=0}^N c_k^2 \sum_{k=0}^N \Big[f(n_k x)-S_{\tau_0(k)}(n_k x)\Big]^2 dx$$

$$\leqq \sum_{k=0}^\infty c_k^2 \sum_{k=0}^\infty \int_0^1 \Big[f(n_k x)-S_{\tau_0(k)}(n_k x)\Big]^2 dx \leqq \sum_{k=0}^\infty c_k^2 \sum_{k=0}^\infty \frac{1}{(k+1)^\alpha} \leqq A\sum_{k=0}^\infty c_k^2$$

Thus we obtain the case $l=-1$ of (3.6).

For a fixed $l(0\leqq l\leqq q-1)$, putting for i=0,1,2,...

$$X_i^{(l)}(x)=\sum_{k=\nu_i+1}^{\nu_{i+1}} c_k[S_{\tau_l(k)}(n_k x)-S_{\tau_{l+1}(k)}(n_k x)], \tag{3.7}$$

where $\{\nu_i\}$ is a steadily increasing sequence of integers defined by

$$\nu_0=0\,,\ \nu_{i+1}=\nu_i+[\frac{2}{\log c}\lambda_l(\nu_i)] \qquad (i=0,1,2,\ldots) \tag{3.8}$$

then $X_i^{(l)}(x)$ is a trigonometric polynomial, and each frequency of its terms lies in the interval $[\tau_{l+1}(\nu_i)n(\nu_i),\ \tau_l(\nu_{i+1})n(\nu_{j+1})]$. Concering these intervals, we have by (1.3), (3.8) and (3.2) the following inequalities.

$$\frac{\tau_{l+1}(\nu_{i+1})n(\nu_{i+1})}{\tau_l(\nu_i)\ n(\nu_i)} \geqq c^{\nu_{i+1}-\nu_i}\frac{1}{\tau_l(\nu_i)} \geqq e>1 \qquad (i=0,1,2,\ldots), \tag{3.9}$$

where $n(k)$ means $n_k(k=1,2,\cdots)$

For any integers M and N such as $0\leqq M<N$, we have, from Lemma 2 and (3.9)

$$\int_0^1 \Big(\sum_{i=M}^N X_{2i}^{(l)}(x)\Big)^2 dx=\sum_{i=M}^N \int_0^1 X_{2i}^{(l)}(x)^2 dx$$

$$\leqq A\sum_{i=M}^N \sum_{k=\nu_{2i}+1}^{\nu_{2i+1}} c_k^2\Big(\log\tau_{l+1}(k)\Big)^{-\alpha} \leqq A\sum_{k=\nu_{2M}+1}^{\infty} c_k^2 \tag{3.10}$$

By (3.1) and (3.10), $\sum X_{2i}^{(l)}(x)$ is the Fourier series of some spuare integrable function, and Lemma 1 shows that

$$\int_0^1 \sup_N \Big(\sum_{i=0}^N X_{2i}^{(l)}(x)\Big)^2 dx \leqq A\sum_{i=0}^\infty \sum_{k=\nu_{2i}+1}^{\nu_{2i+1}} c_k^2 \leqq A\sum_{k=0}^\infty c_k^2 \tag{3.11}$$

Now if we use the similar way concering $\sum X_{2i+1}^{(l)}(x)$, we obtain the corresponding inequality to (3.11). Hence we have

$$\int_0^1 \sup_N \Big(\sum_{i=0}^N X_i^{(l)}(x)\Big)^2 dx \leqq 2\int_0^1 \Big[\sup_N \Big(\sum_{i=0}^N X_{2i}^{(l)}(x)\Big)^2+\sup_N\Big(\sum_{i=0}^N X_{2i+1}(x)\Big)^2\Big]dx \tag{3.12}$$

$$\leqq A\sum_{k=0}^\infty c_k^2$$

Using (3.12), Lemma 2 and the well known Menchoff's lemma, we obtain the following estimations.

$$\int_0^1 Y_l(x)dx=\int_0^1 \sup_N \Big(\sum_{k=0}^N c_k\Big[S_{\tau_l(k)}(n_k x)-S_{\tau_{l+1}(k)}(n_k x)\Big]\Big)^2 dx$$

$$\leqq 2\int_0^1 \sup_N \Big(\sum_{i=0}^N X_i^{(l)}(x)\Big)^2 dx$$

$$+2\int_0^1 \sup_N \sup_{\nu_N<m<\nu_{N+1}} \Big(\sum_{k=\nu_N+1}^m c_k\Big[S_{\tau_l(k)}(n_kx)-S_{\tau_{l+1}(k)}(n_kx)\Big]\Big)^2 dx$$

$$\leqq A\sum_{k=0}^\infty c_k^2+2\sum_{N=0}^\infty \int_0^1 \sup_{\nu_N<m<\nu_{N+1}} \Big(\sum_{k=\nu_N+1}^m c_k\Big[S_{\tau_l(k)}(n_kx)-S_{\tau_{l+1}(k)}(n_kx)\Big]\Big)^2 dx$$

$$\leqq A\sum_{k=0}^\infty c_k^2+2A\sum_{N=0}^\infty \Big(\log(\nu_{N+1}-\nu_N)\Big)^2 \sum_{k=\nu_N+1}^{\nu_{N+1}} c_k^2\Big(\log \tau_{l+1}(k)\Big)^{-\alpha}$$

$$\leqq A\sum_{k=0}^\infty c_k^2+A_c\sum_{N=0}^\infty \Big(\log \lambda_l(\nu_N)\Big)^2 \sum_{k=\nu_N+1}^{\nu_{N+1}} c_k^2\Big(\lambda_{l+1}(k)\Big)^{-\alpha}$$

$$\leqq A\sum_{k=0}^\infty c_k^2+A_c\sum_{N=0}^\infty \sum_{k=\nu_N+1}^{\nu_{N+1}} c_k^2\Big(\log_{l+1}(k+B)\Big)^{-2}\Big(\log_{l+1}(k+B)\Big)^2$$

$$\leqq A_c\sum_{k=0}^\infty c_k^2 \tag{3.13}$$

Lastly we have to estimate $\int_0^1 Y_q(x)\,dx$, but this runs on the same lines as (3.13). Namely, if for $\{\nu_i\}$ defined by putting $l=q$ at (3.8), we define

$$X_i^{(q)}(x)=\sum_{k=\nu_i+1}^{\nu_{i+1}} c_k S_{\tau_q(k)}(n_kx), \qquad (i=0,1,2,\ldots) \tag{3.14}$$

then it is easily seen that

$$\int_0^1 \sup_N \Big(\sum_{i=0}^N X_i^{(q)}(x)\Big)^2 dx \leqq A_c\sum_{k=0}^\infty c_k^2,$$

which corresponds to (3.12), and then by Lemma 2(ii) and the Menchoff's lemma

$$\int_0^1 \sup_N \Big(\sum_{k=0}^N c_k S_{\tau_q(k)}(n_kx)\Big)^2 dx$$

$$\leqq A\int_0^1 \sup_N \Big(\sum_{i=0}^N X_i^{(q)}(x)\Big)^2 dx+A\int_0^1 \sup_N \sup_{\nu_N<m<\nu_{N+1}} \Big(\sum_{k=\nu_N+1}^m c_k S_{\tau_q(k)}(n_kx)\Big)^2 dx$$

$$\leqq A\sum_{k=0}^\infty c_k^2+A\sum_{N=0}^\infty \int_0^1 \sup_{\nu_N<m<\nu_{N+1}} \Big(\sum_{k=\nu_N+1}^m c_k S_{\tau_q(k)}(n_kx)\Big)^2 dx$$

$$\leqq A\sum_{k=0}^\infty c_k^2+A\sum_{N=0}^\infty \Big(\log(\nu_{N+1}-\nu_N)\Big)^2 \sum_{k=\nu_N+1}^{\nu_{N+1}} c_k^2$$

$$\leqq A\sum_{k=0}^\infty c_k^2+A_c\sum_{N=0}^\infty \Big(\log \lambda_q(\nu_N)\Big)^2 \sum_{k=\nu_N+1}^{\nu_{N+1}} c_k^2 \leqq A\sum_{k=0}^\infty c_k^2+A_c\sum_{k=0}^\infty c_k^2\Big(\log_{q+1}(k+B)\Big)^2$$

$$\leqq A_c\sum_{k=0}^\infty c_k^2\Big(\log_{q+1}(k+B)\Big)^2 \tag{3.15}$$

Thus by (3.13) and (3.15) we obtain (3.6).

4. Proof of Theorem 2. In order to prove Theorem 2, we may suppose

$$\sum_{k=0}^{\infty} c_k^2 < +\infty \tag{4.1}$$

Let $\sigma_N^{(\beta)}$ (x) denote the $(C, \beta > 0)$-mean of (1.2), then by (2.3) we have

$$\sigma_N^{(\beta)}(x) = \frac{1}{A_N^{(\beta)}} \sum_{k=0}^{N} A_{N-k}^{(\beta)} c_k f(n_k x) =$$

$$= \frac{1}{A_N^{(\beta)}} \sum_{k=0}^{N} A_{N-k}^{(\beta)} c_k S_{\tau(k)}(n_k x) + \frac{1}{A_N^{(\beta)}} \sum_{k=0}^{N} A_{N-k}^{(\beta)} c_k \left[f(n_k x) - S_{\tau(k)}(n_k x) \right]$$

$$= U_N^{(\beta)}(x) + V_N^{(\beta)}(x)$$

Now we have by (1.4) and (2.3)

$$\int_0^1 \sup_N \left(V_N^{(\beta)}(x) \right)^2 dx \leq \int_0^1 \sup_N \frac{1}{\left(A_N^{(\beta)}\right)^2} \sum_{k=0}^{N} \left(A_{N-k}^{(\beta)} c_k \right)^2 \sum_{k=0}^{N} \left[f(n_k x) - S_{\tau(k)}(n_k x) \right]^2$$

$$\leq \int_0^1 \sup_N \frac{1}{\left(A_N^{(\beta)}\right)^2} \left(A_N^{(\beta)} \right)^2 \sum_{k=0}^{N} c_k^2 \sum_{k=0}^{N} \left[f(n_k x) - S_{\tau(k)}(n_k x) \right]^2$$

$$\leq \sum_{k=0}^{\infty} c_k^2 \sum_{k=0}^{\infty} \int_0^1 \left[f(n_k x) - S_{\tau(k)}(n_k x) \right]^2 dx \leq \sum_{k=0}^{\infty} c_k^2 A \sum_{k=0}^{\infty} \left(\log \tau(k) \right)^{-2\alpha}$$

$$\leq A \sum_{k=0}^{\infty} c_k^2 \sum_{k=0}^{\infty} \frac{1}{(k+1)^{\alpha}} \leq A \sum_{k=0}^{\infty} c_k^2 \tag{4.2}$$

Next if we put

$$X_i(x) = \sum_{k=2^i+1}^{2^{i+1}} c_k S_{\tau(k)}(n_k x), \qquad (i=0,1,2,\ldots) \tag{4.3}$$

then by the similar way as §3, we can verify that there exist some spuare integrable functions $g(x)$ and $h(x)$, such that

$$g(x) \sim \sum_{i=0}^{\infty} X_{2i}(x), \qquad h(x) \sim \sum_{i=0}^{\infty} X_{2i+1}(x),$$

and

$$\int_0^1 \sup_N \left(\sum_{k=1}^{2^N} c_k S_{\tau(k)}(n_k x) \right)^2 dx \leq A \int_0^1 [g(x)^2 + h(x)^2] dx \leq A \sum_{k=0}^{\infty} c_k^2 \tag{4.4}$$

By the definition (2.7) of U_N(x), Lemma 2(ii), and (4.4), we obtain

$$\int_0^1 \sup_N \left(U_{2^N}(x) \right)^2 dx$$

$$\leq 2 \int_0^1 \sup_N \left(U_{2^N}(x) - \sum_{k=0}^{2^N} c_k S_{\tau(k)}(n_k x) \right)^2 dx + 2 \int_0^1 \sup_N \left(\sum_{k=0}^{2^N} c_k S_{\tau(k)}(n_k x) \right)^2 dx$$

$$\leq A \int_0^1 \sup_N \left(\frac{1}{2^N} \sum_{k=0}^{2^N} k c_k S_{\tau(k)}(n_k x) \right)^2 dx + A \sum_{k=0}^{\infty} c_k^2$$

$$\leqq A\sum_{N=0}^{\infty}\frac{1}{2^{2N}}\sum_{k=0}^{2^N}k^2c_k^2+A\sum_{k=0}^{\infty}c_k^2=A\sum_{k=0}^{\infty}k^2c_k^2\sum_{2^N\geqq k}\frac{1}{2^{2N}}+A\sum_{k=0}^{\infty}c_k^2$$

$$\leqq A\sum_{k=0}^{\infty}c_k^2 \tag{4.5}$$

Henceforth our proof runs on the same lines as that of P. Alexits ([5], page 109). However we proceed the proof for sake of completeness of our arguments.

Putting first of all, for $\frac{1}{2}<\varepsilon<1$,

$$\delta_n(x)=U_n^{(\varepsilon)}(x)-U_n^{(\varepsilon+1)}(x)=\frac{1}{(1+\varepsilon)A_n^{(1+\varepsilon)}}\sum_{k=0}^{n}kc_kA_{n-k}^{(\varepsilon)}S_{\tau(k)}(n_kx) \tag{4.6}$$

then we have, by Lemma 2,

$$\sum_{n=0}^{\infty}\int_0^1\delta_{2^n}(x)^2\,dx=\sum_{n=0}^{\infty}\frac{A}{2^{2(1+\varepsilon)n}}\sum_{k=0}^{2^n}k^2c_k^2(2^n-k+1)^{2\varepsilon}$$

$$=A\sum_{k=0}^{\infty}k^2c_k^2\sum_{2^n\geqq k}\frac{(2^n-k+1)^{2\varepsilon}}{2^{2(1+\varepsilon)n}}\leqq A\sum_{k=0}^{\infty}k^2c_k^2\sum_{2^n\geqq k}\frac{1}{2^{2n}}\left(1-\frac{k-1}{2^n}\right)^{2\varepsilon}$$

$$\leqq A\sum_{k=0}^{\infty}k^2c_k^2\sum_{2^n\geqq k}\frac{1}{2^{2n}}\leqq A\sum_{k=0}^{\infty}c_k^2, \tag{4.7}$$

and by Lemma 4, (4.5) and (4.7), we heve

$$\int_0^1\sup_N\left(U_{2^N}^{(\varepsilon)}(x)\right)^2dx\leqq 2\int_0^1\sup_N\left(U_{2^N}^{(\varepsilon)}(x)-U_{2^N}^{(1+\varepsilon)}(x)\right)^2dx+2\int_0^1\sup_N\left(U_{2^N}^{(1+\varepsilon)}(x)\right)^2dx$$

$$\leqq 2\int_0^1\sup_N\left(\delta_{2^N}(x)\right)^2dx+A\int_0^1\sup_N\left(U_N(x)\right)^2dx$$

$$\leqq A\sum_{k=0}^{\infty}c_k^2+A\sum_{k=0}^{\infty}c_k^{2*}\leqq A\sum_{k=0}^{\infty}c_k^2 \tag{4.8}$$

From (2.4), (4.8) and Lemma 3, we have

$$\int_0^1\sup_N\left(U_N^{(\varepsilon)}(x)\right)^2dx\leqq 2\int_0^1\sup_N\left(U_{2^N}^{(\varepsilon)}(x)\right)^2dx+2\int_0^1\sup_N\sup_{2^N<m<2^{N+1}}$$

$$\left(U_m^{(\varepsilon)}(x)-U_{2^N}^{(\varepsilon)}(x)\right)^2dx\leqq A\sum_{k=0}^{\infty}c_k^2+2\sum_{N=0}^{\infty}\int_0^1\sup_{2^N<m<2^{N+1}}\left(U_m^{(\varepsilon)}(x)-U_{2^N}^{(\varepsilon)}(x)\right)^2dx$$

$$\leqq A\sum_{k=0}^{\infty}c_k^2+A\sum_{k=0}^{\infty}c_k^2\leqq A\sum_{k=0}^{\infty}c_k^2 \tag{4.9}$$

Thus from (4.2) and (4.9) we obtain, for $\frac{1}{2}<\varepsilon<1$

$$\int_0^1\sup_N\left(\sigma_N^{(\varepsilon)}(x)\right)^2dx\leqq A\sum_{k=0}^{\infty}c_k^2 \tag{4.10}$$

---

*) We obtain $\int_0^1\sup_m U_m(x)^2dx\leqq A\sum_{k=0}^{\infty}c_k^2$, because of (4.5), Lemma 3 and the following estimations $\int_0^1\sup_m U_m(x)^2dx\leqq 2\int_0^1\sup_n U_{2^n}(x)^2dx+2\int_0^1\sup_n\sup_{2^n<m<2^{n+1}}[U_m(x)-U_{2^n}(x)]^2dx$ $\leqq A\sum c_k^2+2\sum_{n=0}^{\infty}\int_0^1\sup_{2^n<m<2^{n+1}}[U_m(x)-U_{2^n}(x)]^2\,dx\leqq A\sum c_k^2$.

Now if we put, for $\frac{1}{2}<\varepsilon<1$,

$$\int_0^1 \sup_N \frac{1}{N}\sum_{k=0}^N \left(U_k^{(\varepsilon-1)}(x)\right)^2 dx \leqq 2\int_0^1 \sup_N \left(\frac{1}{N}\sum_{k=0}^N \left[U_k^{(\varepsilon-1)}(x)-U_k^{(\varepsilon)}(x)\right]^2\right)dx$$

$$+2\int_0^1 \sup_N \frac{1}{N}\sum_{k=0}^N (U_k^{(\varepsilon)}(x))^2 dx \equiv P+Q,$$

then by (4.9) we obtain

$$Q\leqq 2\int_0^1 \sup_N \left(U_N^{(\varepsilon)}(x)\right)^2 dx \leqq A\sum_{k=0}^\infty c_k^2 \tag{4.11}$$

Moreover if we put

$$\eta_N(x)=\frac{1}{N}\sum_{n=0}^N \left[U_n^{(\varepsilon-1)}(x)-U_n^{(\varepsilon)}(x)\right]^2,$$

then from Lemma 2, we have the following estimations

$$\sum_{N=0}^\infty \int_0^1 \eta_{2^N}(x)\,dx=\sum_{N=0}^\infty \frac{1}{2^N}\sum_{n=0}^{2^N}\int_0^1 \left[U_n^{(\varepsilon-1)}(x)-U_n^{(\varepsilon)}(x)\right]^2 dx$$

$$=\sum_{N=0}^\infty \frac{1}{2^N}\sum_{n=0}^{2^N}\int_0^1\left(\frac{1}{\varepsilon A_n^{(\varepsilon)}}\sum_{k=0}^n A_{n-k}^{(\varepsilon-1)} kc_k\, S_{\tau(k)}(n_k x)\right)^2 dx$$

$$\leqq A\sum_{N=0}^\infty \frac{1}{2^N}\sum_{n=0}^{2^N}\frac{1}{\varepsilon^2\left(A_n^{(\varepsilon)}\right)^2}\sum_{k=0}^n\left(A_{n-k}^{(\varepsilon-1)}\right)^2 k^2c_k^2$$

$$=A\sum_{N=0}^\infty \frac{1}{2^N}\sum_{k=0}^{2^N} k^2c_k^2 \sum_{k\leqq n\leqq 2^N}\left(A_{n-k}^{(\varepsilon-1)}\right)^2\left(A_n^{(\varepsilon)}\right)^{-2}$$

$$\leqq A\sum_{N=0}^\infty \frac{1}{2^N}\sum_{k=0}^{2^N} k^2c_k^2\frac{1}{k}=A\sum_{k=0}^\infty kc_k^2\sum_{2^N\geqq k}\frac{1}{2^N}\leqq A\sum_{k=0}^\infty c_k^2 \tag{4.12}$$

Now we easily estimate P, namely, by (4.12)

$$P\leqq\int_0^1 \sup_N \eta_N(x)\,dx\leqq\int_0^1 \sup_N \eta_{2^N}(x)\,dx+\int_0^1 \sup_N \sup_{2^N<m<2^{N+1}} |\eta_m(x)-\eta_{2^N}(x)|\,dx$$

$$\leqq\sum_{N=0}^\infty\int_0^1 \eta_{2^N}(x)\,dx+\sum_{N=0}^\infty\int_0^1 (2\,\eta_{2^{N+1}}(x)+\eta_{2^N}(x))\,dx\leqq A\sum_{k=0}^\infty c_k^2 \tag{4.13}$$

From (4.11) and (4.13), we have

$$\int_0^1 \sup_N \frac{1}{N}\sum_{k=0}^N\left(U_k^{(\varepsilon-1)}(x)\right)^2 dx\leqq A\sum_{k=0}^\infty c_k^2 \tag{4.14}$$

After these preparations we can now easily verify Theorem 2. Let $\frac{1}{2}>\beta>0$, then (it is clear that, without loss of generality, we may suppose $0<\beta<\frac{1}{2}$)

$$U_n^{(\beta)}(x)=\frac{1}{A_n^{(\beta)}}\sum_{k=0}^n A_{n-k}^{(\beta-\varepsilon)}\,A_k^{(\varepsilon-1)}\,U_k^{(\varepsilon-1)},$$

where ε satisfies $\frac{1}{2}<\varepsilon<\frac{1}{2}+\beta<1$, and by (4,14),

$$\int_0^1 \sup_N \left(U_N^{(\beta)}(x)\right)^2 dx \leqq \int_0^1 \sup_N \frac{1}{\left(A_N^{(\beta)}\right)^2} \sum_{k=0}^N \left(A_{N-k}^{(\beta-\varepsilon)} A_k^{(\varepsilon-1)}\right)^2 \sum_{k=0}^N \left(U_k^{(\varepsilon-1)}\right)^2 dx$$

$$\leqq \int_0^1 \sup_N \left(\frac{1}{N} \sum_{k=0}^N \left(U_k^{(\varepsilon-1)}(x)\right)^2\right) \sup_N \frac{N}{\left(A_N^{(\beta)}\right)^2} \sum_{k=0}^N \left(A_{N-k}^{(\beta-\varepsilon)} A_k^{(\varepsilon-1)}\right)^2 dx$$

$$\leqq A \sum_{k=0}^\infty c_k^2 \sup_N \frac{1}{N^{2\beta-1}} \sum_{k=0}^N \left(A_{N-k}^{(\beta-\varepsilon)} A_k^{(\varepsilon-1)}\right)^2$$

$$= A \sum_{k=0}^\infty c_k^2 \sup_N \frac{1}{N^{2\beta-1}} N^{2\beta-1} \leqq A \sum_{k=0}^\infty c_k^2$$

This and (4.2) show the truth of Theorem 2.

References

[1] L. Kantorovitch, Some Theorem on the Almost Everywhere Convergence, Comptes Rendus Acad. Sci. URSS., 14(1937), pp. 537-540.

[2] G. Sunouchi and S. Yano, Convergence and Summability of Orthogonal Series, Proc. Jap. Acad., 26(1950), pp. 10-16.

[3] M. Kac, R. Salem and A. Zygmund, A Gap Theorem, Trans. Amer. Math. Soc., 63(1948), pp. 235-243.

[4] A. Zygmund, Trigonometsic Series (Warszawa-Lwów, (1935), p. 252).

[5] G. Alexits, Convergence Problems of Orthogonal Series, (1961), p. 125.

[6] ————, ibid. p. 52 and p. 54.