

## A Note on Convergence of Gap Series

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**1. Introduction.** In the present note we shall assume that  $f(t)$  is an integrable function with period 1, such that

$$\int_0^1 f(t) dt = 0, \quad \int_0^1 f(t)^2 dt = 1,$$

and  $\{n_k\}$  is a sequence of positive integers satisfying  $n_{k+1}/n_k > c > 1 (k=1, 2, \dots)$ . M. Kac, R. Salem and A. Zygmund [1] proved the following theorem;

**Theorem A.** If, for  $\alpha > 1$ ,  $f(t)$  satisfies

$$(1.1) \quad \left( \int_0^1 |f(t) - S_n(t)|^2 dt \right)^{1/2} = O\left( (\log n)^{-\alpha} \right)$$

as  $n \rightarrow \infty$ , then

$$(1.2) \quad \sum c_k f(n_k t)$$

converges a. e., provided that

$$(1.3) \quad \sum c_k^2 (\log k)^2 < \infty,$$

where  $S_n(t)$  means the  $n$ -th partial sum of the Fourier series of  $f(t)$ . On the other hand if, for  $0 < \alpha \leq 1$ ,  $f(t)$  satisfies (1.1), then

$$(1.2') \quad \sum \frac{1}{k^\beta} f(n_k t)$$

converges a. e., provided that

$$(1.3') \quad \beta > 1 - \frac{\alpha}{2}.$$

In Theorem 1 and Theorem 2 of this note, we shall prove that (1.3) and (1.3') may be replaced by more weaker restrictions respectively. On the other hand S. Izumi [2] obtained a result concerning the special case  $\beta = 1$ . Our Theorem 2 (iii) contains a generalization of the Izumi's result.

**2. Theorem 1.** If, for  $\alpha > 1$ ,  $f(t)$  satisfies (1.1) and for some non-negative integer  $\sigma \geq 0$ ,

$$(2.1) \quad \sum c_k^2 (\log_{\sigma+1} k)^2 < \infty$$

holds, then (1.2) converges a. e., where  $\log_{\sigma}$  means the  $\sigma$  times iteration of the logarithm

For the proof of this theorem, we need two lemmas.

**Lemma 1.** If, for  $\alpha > 0$ ,  $f(t)$  satisfies (1.1), then for any positive integers  $M$  and  $N$  such as  $M < N$ , and for any non-decreasing sequence  $\{\tau(k)\}$  of integers, we have

$$(2.2) \quad \int_0^1 \left| \sum_M^N c_k \{f(n_k t) - S_{\tau(k)}(n_k t)\} \right|^2 dt \leq F(M, N; \alpha) \sum_M^N c_k^2 (\log \tau(k))^{-\alpha}$$

where  $F(M, N; \alpha)$  is defined by

$$(2.3) \quad \begin{aligned} F(M, N; \alpha) &= O((N-M)^{1-\alpha}) \quad \text{for } 0 < \alpha < 1, \\ &= O(\log(N-M)) \quad \text{for } \alpha = 1, \\ &= O(1) \quad \text{for } \alpha > 1, \end{aligned}$$

**Proof.** If we put, for simplicity of writing,  $\varphi_n(t) = f(t) - S_n(t)$ , then it is familiar that for  $j < k$ ,

$$\int_0^1 \varphi_{\tau(j)}(n_j t) \varphi_{\tau(k)}(n_k t) dt = O((k-j)^{-\alpha} (\log \tau(k))^{-\alpha})$$

Now from the above relation, we have

$$\begin{aligned} & \int_0^1 \left| \sum_M^N c_k \varphi_{\tau(k)}(n_k t) \right|^2 dt = \sum_M^N c_k^2 \int_0^1 \varphi_{\tau(k)}(n_k t)^2 dt \\ & + \sum_{k \neq j} c_k c_j \int_0^1 \varphi_{\tau(k)}(n_k t) \varphi_{\tau(j)}(n_j t) dt \\ & = \sum_M^N c_k^2 O(\log \tau(k))^{-2\alpha} + 2 \sum_{j=M}^{N-1} \sum_{k=j+1}^N c_k c_j O((k-j)^{-\alpha} (\log \tau(k))^{-\alpha}) \\ & = \sum_M^N c_k^2 O(\log \tau(k))^{-2\alpha} + 2 \sum_{j=M}^{N-1} \sum_{r=1}^{N-j} c_{j+r} c_j O(r^{-\alpha} (\log \tau(j+r))^{-\alpha}) \\ & < O(1) \sum_M^N c_k^2 (\log \tau(k))^{-2\alpha} + O(1) \sum_{r=1}^{N-M} \frac{1}{r^{\alpha}} \left( \sum_M^N c_k^2 (\log \tau(k))^{-\alpha} \right) \end{aligned}$$

Hence we obtain the lemma.

**Lemma 2.** Under the hypotheses of Lemma 1, we have

$$(2.4) \quad \int_0^1 \left| \sum_M^N c_k S_{\tau(k)}(n_k t) \right|^2 dt \leq F(M, N; \alpha) \sum_M^N c_k^2.$$

The Proof is similar as that of Lemma 1.

**Remark.** In Lemma 1, we may replace  $\varphi_{\tau(k)}(n_k t)$  by  $\{S_{N(k)}(n_k t) - S_{M(k)}(n_k t)\}$ , where  $M(k)$  and  $N(k)$  are both non-decreasing sequence of integers such as  $M(k)$

$<N(k)$ , and we have the following inequality,

$$\int_0^1 \left| \sum_{\frac{N}{M}}^N c_k \{S_{N(k)}(n_k t) - S_{M(k)}(n_k t)\} \right|^2 dt \leq F(M, N; \alpha) \sum_{\frac{N}{M}}^N c_k^2 (\log M(k))^{-\alpha}$$

**Proof of Theorem 1:** Putting, for  $k=1, 2, \dots$ ,

$$(2.5) \quad \tau_0(k) = [\exp \lambda_0(k)] \text{ and } \lambda_0(k) = \sqrt{k},$$

then we have

$$\sum |c_k| \int_0^1 |\varphi_{\tau_0(k)}(n_k t)| dt < \sum |c_k| O\left((\log \tau_0(k))^{-\alpha}\right) < \infty.$$

This formula indicates the almost everywhere convergence of

$$(2.6) \quad \sum c_k \varphi_{\tau_0(k)}(n_k t),$$

Now putting, for  $q=1, 2, 3, \dots$

$$(2.7) \quad \tau_q(k) = [\exp \lambda_q(k)] \text{ and } \lambda_q(k) = (\log_q k)^{2/\alpha},$$

then for some  $k_0$ ,  $\lambda_q(k)$  is non-decreasing as  $k \geq k_0 = k_0(q)$ , and we have, for each  $q=0, 1, 2, \dots$ , the almost everywhere convergence of

$$(2.8) \quad \sum c_k \{S_{\tau_q(k)}(n_k t) - S_{\tau_{q+1}(k)}(n_k t)\}.$$

For the proof of this, we consider a sequence of trigonometric polynomials

$$(2.9) \quad X_k(t) = \sum_{l=\nu_k}^{\nu_{k+1}-1} c_l \{S_{\tau_q(l)}(n_l t) - S_{\tau_{q+1}(l)}(n_l t)\}$$

where  $\{\nu_k\}$  is an increasing sequence of integers such that, for  $k_0(q)$

$$(2.10) \quad \nu_{k_0} = \text{some constant } A, \nu_{k+1} = \nu_k + \left[ \frac{2}{\log c} \lambda_q(\nu_k) \right]$$

Since the largest frequency of  $X_{k-1}(t)$  is  $\tau_q(\nu_k) n(\nu_k)$  and the smallest frequency of  $X_{k+1}(t)$  is  $\tau_{q+1}(\nu_{k+1}) n(\nu_{k+1})$ , and moreover we have from the definition of  $\{\nu_k\}$

$$(2.11) \quad \frac{\tau_{q+1}(\nu_{k+1}) n(\nu_{k+1})}{\tau_q(\nu_k) n(\nu_k)} > e > 1 \quad (k \geq k_0),$$

where  $n(k)$  means  $n_k (k=1, 2, \dots)$ , so both sequences  $\{X_{2k}(t); 2k \geq k_0\}$  and  $\{X_{2k-1}(t); 2k-1 \geq k_0\}$  are orthogonal respectively, and by Remark of Lemma 1,

$$\begin{aligned} \int_0^1 |\sum X_{2k}(t)|^2 dt &= \sum \int_0^1 |X_{2k}(t)|^2 dt \\ &\leq \sum_k \sum_{l=\nu_{2k}}^{\nu_{2k+1}-1} c_l^2 (\log \tau_{q+1}(l))^{-\alpha} < \sum c_l^2 < \infty. \end{aligned}$$

From this and (2.11),  $\sum X_{2k}(t)$  is the Fourier series of some square integrable function and it converges a.e.. By the same way, we obtain the almost everywhere convergence of  $\sum X_{2k-1}(t)$  and consequently  $\sum X_k(t)$ . Now the convergence of (2.8) follows from

$$(2.12) \quad \sum_k \int_0^1 \max_{\nu_k < m < \nu_{k+1}} \left| \sum_{l=\nu_k}^m c_l \{S_{\tau_q(l)}(n_l t) - S_{\tau_{q+1}(l)}(n_l t)\} \right|^2 dt < \infty.$$

But this (2.12) is easily verified by use of the Menchov's well known devices. The left hand side of (2.12) is less than by (2.7)

$$\begin{aligned} &\leq \sum_k \left( \log (\nu_{k+1} - \nu_k) \right)^2 \int_0^1 |X_k(t)|^2 dt \\ &\leq O(1) \sum_k \left( \log \lambda_q(\nu_k) \right)^2 \sum_{l=\nu_k}^{\nu_{k+1}-1} c_l^2 \left( \log \tau_{q+1}(l) \right)^{-\alpha} \\ &\leq O(1) \sum_k \sum_{l=\nu_k}^{\nu_{k+1}-1} c_l^2 \lambda_{q+1}(l)^{-\alpha} \left( \log \lambda_q(l) \right)^2 < O(1) \sum c_k^2 \end{aligned}$$

As a conclusion of the above arguments, we obtain the following

$$(2.13) \quad \left( \begin{array}{l} \text{Under the hypothesis (1.1) and } \sum c_k^2 < \infty, \text{ whenever } q \text{ is 0 or a} \\ \text{positive integer, then } \sum_{k=k_0(q)}^{\infty} c_k \{ f(n_k t) - S_{\tau_q(k)}(n_k t) \} \\ \text{converges a.e.} \end{array} \right)$$

Lastly if we prove, for some  $q$ , the convergence of

$$(2.14) \quad \sum c_k S_{\tau_q(k)}(n_k t),$$

then we obtain Theorem 1. Using Lemma 2 instead of Lemma 1, we can easily lead, by the similar methods as the above arguments, for each  $q=0, 1, 2, \dots$  the almost everywhere convergence of

$$(2.15) \quad \sum_k \sum_{l=\nu_k}^{\nu_{k+1}-1} c_l S_{\tau_q(l)}(n_l t),$$

and, have, for each  $q=0, 1, 2, \dots$

$$\begin{aligned} &\sum_k \int_0^1 \max_{\nu_k < m < \nu_{k+1}} \left| \sum_{l=\nu_k}^m c_l S_{\tau_q(l)}(n_l t) \right|^2 dt \\ (2.16) \quad &\leq \sum_k \left( \log (\nu_{k+1} - \nu_k) \right)^2 \int_0^1 \left| \sum_{l=\nu_k}^{\nu_{k+1}-1} c_l S_{\tau_q(l)}(n_l t) \right|^2 dt \\ &\leq \sum_k \left( \log \lambda_q(\nu_k) \right)^2 \sum_{l=\nu_k}^{\nu_{k+1}-1} c_l^2 \leq O(1) \sum_l c_l^2 (\log_{q+1} l)^2 \end{aligned}$$

From (2.1), the last series converges when  $q=\sigma$ . Thus by (2.13), (2.15) and (2.16) we obtain the theorem.

**3. Theorem 2.** If  $f(t)$  satisfies (1.1), then we have,

- (i) for  $0 < \alpha < \frac{1}{2}$  and  $\beta > 1 - \alpha$ , (1.2') converges a.e.,
- (ii) for  $\frac{1}{2} \leq \alpha < 1$  and  $\beta > \frac{1}{2}$ , (1.2') converges a.e.,
- (iii) for  $\alpha = 1$  and  $\beta > \frac{1}{2}$

$$\sum \frac{f(n_k t)}{\sqrt{k} (\log k)^\beta}$$

converges a. e.

**Proof of (i).** If  $\alpha$  and  $\beta$  satisfy the hypothesis of (i), then we have for each  $q=0, 1, 2, \dots$

$$\frac{\beta}{1-\alpha} > \beta \frac{2-\alpha'^q-\alpha'^{q+1}}{1-\alpha'^{q+1}},$$

where  $\alpha'=(1-\alpha)/\alpha$ , so we determine a number  $r$  such as for each  $q=0, 1, 2, \dots$ ,  $\sigma$  (where  $\sigma$  is an arbitrary but fixed number)

$$(3.1) \quad r > 1, \quad \frac{\beta}{1-\alpha} > r > \beta \frac{2-\alpha'^q-\alpha'^{q+1}}{1-\alpha'^{q+1}},$$

and we define for  $q=0, 1, 2, \dots, \sigma$

$$(3.2) \quad \lambda_0 = k^{(r-\beta)/\alpha}, \lambda_{q+1}(k) = k^{(r-2\beta)/\alpha} \lambda_q(k)^{\alpha'}, \tau_q(k) = [\exp \lambda_q(k)].$$

From (3.2)

$$\lambda_q(k) = k^{\frac{r-2\beta}{\alpha} \frac{1-\alpha'^q}{1-\alpha'} + \frac{r-\beta}{\alpha} \alpha'^q}$$

holds, and for each  $q$ ,  $\lambda_q(k)$  is non-decreasing as  $k$  tends to  $+\infty$ , and for each  $k > 1$ , and  $q=0, 1, 2, \dots, \sigma-1$ ,  $\lambda_q(k) \geq \lambda_{q+1}(k)$ ,

Evidently from (3.2)

$$\sum \frac{1}{l^\beta} \int_0^1 |\varphi_{\tau_0(l)}(nlt)| dt < \infty.$$

follows, and this shows the almost everywhere convergence of

$$(3.3) \quad \sum \frac{1}{l^\beta} \{f(nlt) - S_{\tau_0(l)}(nlt)\}.$$

Using the same formulae as (2.9), (2.10) and (2.11), then from Lemma 2, we have

$$\begin{aligned} & \int_0^1 \left| \sum_k X_{2k}(t) \right|^2 dt = \sum_k \int_0^1 |X_{2k}(t)|^2 dt \\ & \leq \sum_k \frac{\nu_{2k+1}^{-1}}{\sum_{l=\nu_{2k}} \frac{1}{l^{2\beta}}} \frac{1}{\lambda_{q+1}(l)^\alpha} (\nu_{2k+1} - \nu_{2k})^{1-\alpha} \\ & \leq \sum_k \frac{\nu_{2k+1}^{-1}}{\sum_{l=\nu_{2k}} \frac{1}{l^{2\beta}}} \frac{\lambda_q(l)^{1-\alpha}}{l^{r-2\beta} \lambda_q(l)^{\alpha\alpha'}} = \sum \frac{1}{l^r} < \infty. \end{aligned}$$

This shows the almost everywhere convergence of  $\sum X_{2k}(t)$ , and by the similar way, we have the convergence of  $\sum X_{2k+1}(t)$  and consequently  $\sum X_k(t)$ .

Now we shall prove for each  $q=0, 1, 2, \dots, \sigma$ .

$$(3.4) \quad \sum_k \int_0^1 \max_{\nu_k < m < \nu_{k+1}} \left| \sum_{l=\nu_k}^m \frac{1}{l^\beta} \{S_{\tau_q(l)}(nlt) - S_{\tau_{q+1}(l)}(nlt)\} \right|^2 dt < \infty$$

holds. For the left hand side of (3.4) is less than

$$\begin{aligned} & \leq \sum_k \log(\nu_{k+1} - \nu_k) \frac{\log(\nu_{k+1} - \nu_k)}{\sum_{v=0}^{\nu_{k+1} - \nu_k}} \frac{(\nu_{k+1} - \nu_k)^{2-v}}{\sum_{u=0}^{\nu_{k+1} - \nu_k}} \\ & \int_0^1 \left| \sum_{l=\nu_k+u2^v}^{\nu_k+(u+1)2^v-1} \frac{1}{l^\beta} \{S_{\tau_q(l)}(nlt) - S_{\tau_{q+1}(l)}(nlt)\} \right|^2 dt \end{aligned}$$

$$\begin{aligned} &\leq \sum_k \log \lambda_q(\nu_k) \sum_{v=0}^{\log \lambda_q(\nu_k)} \sum_{u=0}^{(\nu_{k+1}-\nu_k)2^{-v}} \sum_{l=\nu_k+u2^v}^{\nu_k+(u+1)2^v-1} \frac{1}{l^{2\beta}} \frac{1}{\lambda_{q+1}(l)^\alpha} 2^{v(1-a)} \\ &\leq \sum_l \frac{\log \lambda_q(l) \lambda_q(l)^{1-a}}{l^{2\beta} l^{r-2\beta} \lambda_q(l)^{\alpha a'}} = \sum \frac{\log l}{l^r} < \infty. \end{aligned}$$

Thus by (3.3) and (3.4) we obtain that

$$(3.5) \quad \sum \frac{1}{l^\beta} \{f(n_l t) - S_{\tau_\sigma(l)}(n_l t)\}$$

converge a. e.

Now if we repeat nearly the same estimations as (2.15) and (2.16), we can conclude that the almost everywhere convergence of

$$(3.6) \quad \sum \frac{1}{l^\beta} S_{\tau_\sigma(l)}(n_l t)$$

holds, provided that

$$(3.7) \quad \sum \frac{\log l}{l^{2\beta} l^{-(1-a)} \left( \frac{r-2\beta}{a} \frac{1-a'\sigma}{1-a'} + \frac{r-\beta}{a} a'\sigma \right)} < \infty.$$

For the proof of this, putting

$$X_k(t) = \sum_{l=\nu_k}^{\nu_{k+1}-1} \frac{1}{l^\beta} S_{\tau_\sigma(l)}(n_l t).$$

and

$$\nu_{k+1} = \nu_k + \left[ \frac{2}{\log c} \lambda_\sigma(\nu_k) \right], \nu_1 = \text{some constant } A,$$

then we have

$$\begin{aligned} &\int_0^1 \left| \sum_k X_{2k}(t) \right|^2 dt = \sum_k \int_0^1 |X_{2k}(t)|^2 dt \\ &= \sum_k (\nu_{2k+1} - \nu_{2k})^{1-a} \sum_{\nu_{2k}}^{\nu_{2k+1}-1} \frac{1}{l^{2\beta}} \leq \sum_k O\left(\lambda_\sigma(\nu_{2k})\right)^{1-a} \sum_{\nu_{2k}}^{\nu_{2k+1}-1} \frac{1}{l^{2\beta}} \\ &= O(1) \sum \frac{1}{l^{2\beta}} \lambda_\sigma(l)^{1-a}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &\sum_k \int_0^1 \max_{\nu_k < m < \nu_{k+1}} \left| \sum_{l=\nu_k}^m \frac{1}{l^\beta} S_{\tau_\sigma(l)}(n_l t) \right|^2 dt \\ &\leq \sum_k \log(\nu_{k+1} - \nu_k) \sum_{v=0}^{\log(\nu_{k+1} - \nu_k)} \sum_{u=0}^{(\nu_{k+1} - \nu_k)2^{-v}} \int_0^1 \left| \sum_{l=\nu_k+u2^v}^{\nu_k+(u+1)2^v-1} \frac{1}{l^\beta} S_{\tau_\sigma(l)}(n_l t) \right|^2 dt \\ &\leq \sum_k \log \lambda_\sigma(\nu_k) \sum_v \sum_u \sum_t \frac{1}{l^{2\beta}} 2^{v(1-a)} = \sum_k \log \lambda_\sigma(\nu_k) (\nu_{k+1} - \nu_k)^{1-a} \sum_{l=\nu_k}^{\nu_{k+1}-1} \frac{1}{l^{2\beta}} \\ &= \sum_k \sum_{l=\nu_k}^{\nu_{k+1}-1} \frac{1}{l^{2\beta}} \lambda_\sigma(l)^{1-a} \log \lambda_\sigma(l) = \sum_k \sum_{l=\nu_k}^{\nu_{k+1}-1} \frac{\log l}{l^{2\beta}} l \left( \frac{r-2\beta}{a} \frac{1-a'\sigma}{1-a'} + \frac{r-\beta}{a} a'\sigma \right) (1-a) \end{aligned}$$

These estimations indicate that (3.7) implies the convergence of (3.6).

If we take a sufficiently large  $\sigma$ , then by (3.1)

$$(3.8) \quad 2\beta - (1 - \alpha) \left( \frac{r - 2\beta}{\alpha} \frac{1 - \alpha'^{\sigma}}{1 - \alpha'} + \frac{r - \beta}{\alpha} \alpha'^{\sigma} \right) > 1,$$

consequently (3.7) holds, and (3.6) converges a. e.. From (3.5) and (3.6) the proof of (i) holds.

**Proof of (ii).** For  $\alpha$  and  $\beta$  satisfying the hypotheses of (ii), but  $\alpha \neq \frac{1}{2}$ \*) we put

$$(3.9) \quad r > 1, \quad \beta + \frac{r}{2} > 1,$$

and we define

$$(3.10) \quad \begin{aligned} \lambda_0(k) &= k^{r/2\alpha}, \quad \lambda_{q+1}(k) = \lambda_0(k)^{\alpha'^{q+1}}, \\ \tau_q(k) &= [\exp \lambda_q(k)]. \end{aligned}$$

The methods of the proof of (ii) are similar as that of (i), so we write only conclusions, that is,

[A] for any  $q=0, 1, 2, \dots$

$$(3.11) \quad \sum_{l=1}^{\infty} \frac{1}{l^{\beta}} \{f(n_l t) - S_{\tau_q(l)}(n_l t)\}$$

converges a.e., provided  $2\beta > 1$ ,

[B] for any  $q=0, 1, 2, \dots$

$$(3.12) \quad \sum_{l=1}^{\infty} \frac{1}{l^{\beta}} S_{\tau_q(l)}(n_l t)$$

converges a.e., provided

$$(3.13) \quad 2\beta - \frac{r}{2} \alpha'^{q+1} > 1.$$

Since  $0 < \alpha' < 1$  holds in our case, for sufficiently large  $q$ , (3.13) is satisfied and we obtain the proof of (ii).

**Proof of (iii).** If we put  $r > 2\beta$ ,  $\lambda_0(k) = k^{r/2}$ ,  $\lambda_1(k) = (\log k)^{r-2\beta} (\log k)^3$ ,  $\lambda_2(k) = (\log k)^{r-2\beta} (\log \log k)^3$ , and  $\tau_q(k) = [\exp \lambda_q(k)]$  ( $q=0, 1, 2$ ), then we can also verify the validity of (iii).

**References**

- (1) M. Kac, R. Salem and A. Zygmund, A Gap Theorem, Trans. Amer. Math. Soc., 63 (1948).
- (2) S. Izumi, On the Strong Law of Large Numbers and Gap Series, Tōhoku Math. Jour., 3(1951).

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\*) For the special case  $\alpha = \frac{1}{2}$  and  $\beta > \frac{1}{2}$ , we put  $\varepsilon \leq (2\beta - 1)$ , then there exists, for a given  $r > 1$ , a positive integer  $\sigma$  such that  $\sigma\varepsilon < r < (\sigma + 1)\varepsilon$ . If we define  $\lambda_q(k) = k^{r-\varepsilon q}$ ,  $\tau_q(k) = [\exp \lambda_q(k)]$  for  $q=0, 1, \dots, \sigma$ , then we obtain the result by the similar way as the general case.