On an Isoperimetric Sequence

By

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1. Introduction.

Let K be a convex body in the three dimensional Euclidean space E_3. Connecting with it various quantities can be defined. These quantities, in general, are functionals which depend upon the shape and size of the convex body K. At this time, it is proposed to examine the functional properties of these quantities which depend upon the shape of the body K. Now, if we denote by $\{K[S_0]\}$ a class of all convex bodies with a given surface area S_0 ($0<S_0<\infty$) in E_3 and by $V(K[S_0])$ the volume of a member $K[S_0]$, the volume $V(K[S_0])$ is a functional which depends upon the shape of the body $K[S_0]$ only. In connection with this functional property of the volume, all efforts to solve the *isoperimetric problem* were to prove that the sphere is the one having the greatest volume. *The purpose of this paper is to examine the functional property of the volume* $V(K[S_0])$ *which depends upon the shape of a member* $K[S_0]$ *of the class* $\{K[S_0]\}$.

For the purpose, in §2 we define an *isoperimetric coefficient* of a convex body K in E_3 and study the properties of it. In §3, we shall prove that the isoperimetric coefficient of the parallel sequence to a convex body is a monotone increasing function. In §4, we shall define a *similitude-indicatrix sequence* to a parallel sequence of convex bodies. By the aid of the similitude-indicatrix sequence, we shall study the functional relation between the isoperimetric coefficient and shape of a convex body, and prove that *the superior limiting figures of the parallel sequences to any convex bodies are always the sphere and of all convex bodies in the three dimensional Euclidean space* E_3, *the sphere has the greatest isoperimetric coefficient*. In the end, with the object of verifying the functional relation between the shape and volume of a convex body, we shall give an example of the *isoperimetric sequence,* say $\{\mathfrak{K}(t): 0<t<\infty\}$, which is defined as follows: it is a sub-class of the class $\{K[S_0]\}$ and the correspondence between a member $\mathfrak{K}(t)$ of the sequence and the parameter t ($0<t<\infty$) is one to one and continuous. Further it satisfies the following conditions:

(i) *If* $t_1<t_2$, $V(\mathfrak{K}(t_1))<V(\mathfrak{K}(t_2))$.

(ii) *If* $t\to+0$, $\mathfrak{K}(t)$ *converges to a closed convex set of area* $S_0/2$ *in a plane.*[1)]

(iii) *If* $t\to+\infty$, $\mathfrak{K}(t)$ *converges to the sphere of radius* $\sqrt{S_0/4\pi}$.

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1) We may substitute a straight line for the closed convex set in a plane.

2. The isoperimetric coefficient.

When K is a convex body in E_3, the volume, surface area, integral of mean curvature, total mean curvature and characteristic[2] of the convex body K we denote by $V(K)$, $S(K)$, $M(K)$, $M^*(K)$ and $\kappa(K)$ respectively. Then, *in generall, $V(K)$, $S(K)$, $M(K)$* and *$M^*(K)$ are functionals which depend upon the shape and size of the body K, but the characteristic $\kappa(K)$ is a functional depending on the shape of K only.*

On the other hand, we have a following set of isoperimetric inequalities:

$$S(K)^3 - 36\pi V(K)^2 \geqq 0, \tag{1}$$

$$S(K)^3 - 9\kappa(K) V(K)^2 \geqq 0, \tag{2}$$

$$S(K)^2 - 3M(K) V(K) \geqq 0 \tag{3}$$

and

$$S(K)^2 - 3M^*(K) V(K) \geqq 0. \tag{4}$$

The equality in (1) *holds if and only if the convex body K is a sphere, the one in* (3) *holds if and only if K is a cap-body of a sphere and the one in* (2) *and* (4) *holds if and only if K is a cap-body of a spehere or a tangential body of a sphere* (*see* [4, PP. 30–31]). Now, let us give the following definition:

DEFINITION. *The isoperimetric coefficient $\chi(K)$ of a convex body K is defined by*

$$\chi(K) = \frac{36\pi V(K)^2}{S(K)^3}. \tag{5}$$

Then the isoperimetric coefficient $\chi(K)$ has the following properties:

(i) *It is bounded and we have*

$$0 < \chi(K) \leqq 1$$

where the equality holds if and only if K is a sphere.

(ii) *For the isoperimetric coefficient $\chi(K)$ and characteristic $\kappa(K)$ of the body K, we have*

$$\kappa(K) \cdot \chi(K) \leqq 4\pi \tag{6}$$

where the equality holds if and only if K is a cap-body of a sphere or a tangential body of a sphere.

(iii) *If K_1 is a similitude of K_2, then we have*

$$\chi(K_1) = \chi(K_2).$$

(iv) *If K_1 is a cap-body of a sphere or a tangential body of a sphere, and the characteristic $\kappa(K_1)$ of K_1 is equal to the one $\kappa(K_2)$ of any other convex body K_2, then we have*

2) See the definition of these quantities in [4, PP. 20–20].

$$\chi(K_1) \geqq \chi(K_2)$$

where the equality holds when K_2 is a cap-body of a sphere or a tangential body of a sphere too.

(v) *If each of the convex bodies K_1 and K_2 is a cap-body of a sphere or a tangential body of a sphere respectively and*

$$\kappa(K_1) < \kappa(K_2),$$

then we have

$$\chi(K_1) > \chi(K_2).$$

The property (**i**) including the condition for the equality is readily showed by the isoperimetric inequality (1). On the other hand, we can give an example of such a sequence of convex bodies K_1, K_2, $\cdots$ as their isoperimetric coefficients converge to zero, that is,

$$\lim_{i\to+\infty} \chi(K_i) = 0 . \tag{7}$$

To show it, first we take a convex figure K_0 in a plane E, that is, a closed bounded convex set with interior points in E. And let $\{K_i\}$ be a sequence of convex bodies in E_3 such that $K_i \to K_0$ as $i \to +\infty$. Then we have the relation (7). For if we denote by F_0 (>0) the plane area of K_0, we have $\lim\limits_{i\to+\infty} S(K_i) = 2F_0 > 0$ and $\lim\limits_{i\to+\infty} V(K_i) = 0$. That is, we have the result (7).

The second property is shown by the inequality (2) and the definition of the isoperimetric coefficient. It gives a relation between the isoperimetric coefficient and the characteristic of a convex body K.

The property (**iii**) is clear by the property of the similitude. The properties (**iv**) and (**v**) can be shown easily by the property (**ii**).

3. The parallel sequence to a convex body.

If K is a convex body with inradius r ($0<r<\infty$), we can define its interior parallel sequence $\{K(t) : 0 \leqq t \leqq r\}$[3] and its exterior parallel sequence[4] $\{K(t) : r<t<\infty\}$. Let us combine together these two sequences and call the thus obtained sequence *the parallel sequence to a convex body* K and denote the sequence by $\{K(t) : 0\leqq t<\infty\}$. Further, if we denote by $\chi(t)$ the isoperimetric coefficient of a member $K(t)$ ($0<t<\infty$) of the sequence, we can express it as follows,

$$\chi(t) = \frac{36\pi V(t)^2}{S(t)^3}, \qquad 0<t<\infty.$$

Here we have the following differential formulas by [4, P. 25] :

3) See the definition in [4, PP. 2–3 and 22–23].

4) See the definition in [2, PP. 17–19].

The volume $V(t)$ is differentiable in $0<t<\infty$ *and we have*

$$\frac{dV(t)}{dt}=S(t), \qquad 0<t<\infty.$$

The surface area $S(t)$ is differentiable in $r<t<\infty$ and we have

$$\frac{dS(t)}{dt}=2M(t), \qquad r<t<\infty.$$

In case of the interior parallel sequence, that is, in $0<t\leqq r$, $S(t)$ *is differentiable at any value t except for the critical values and we have*

$$\frac{dS(t)}{dt}=2M^*(t), \qquad 0<t\leqq r.$$

At the critical value ρ, $S(t)$ is one-sided differentiable and we have

$$\lim_{\Delta t\to -0}\frac{S(\rho+\Delta t)-S(\rho)}{\Delta t}=2M^*(\rho-0)\geqq\lim_{\Delta t\to +0}\frac{S(\rho+\Delta t)-S(\rho)}{\Delta t}=2M^*(\rho+0)$$

where the equality holds only when the surface of $K(t)$ contains no flat-parts which converge to rectilinear edges of $K(\rho)$ as $t\to\rho+0$.

Here, let us call "*M^*–critical value*" the value ρ of the parameter t which corresponds to the discontinuity of $M^*(t)$, that is, $M^*(\rho-0)>M^*(\rho+0)$.

Then, by these formulas and the inequalities (3) and (4) *we have in* $r<t<\infty$

$$(8_1)\qquad \frac{d\chi(t)}{dt}=\frac{72\pi V(t)}{S(t)^4}(S(t)^2-3M(t)V(t))\geqq 0$$

where the equality holds at the present time $r<t<\infty$ if and only if $K(t)$ is a sphere.

In case of the interior parallel sequence we have at any value t $(0<t\leqq r)$ except for M^* -critical values,

$$(8_2)\qquad \frac{d\chi(t)}{dt}=\frac{72\pi V(t)}{S(t)^4}(S(t)^2-3M^*(t)V(t))\geqq 0$$

where the equality holds if and only if $K(t)$ is a cap-body of a sphere, or a tangential body of a sphere.

Further at the M^*-critical value ρ $(0<\rho<r)$ we have

$$(8_3)\qquad \lim_{\Delta t\to +0}\frac{\chi(\rho+\Delta t)-\chi(\rho)}{\Delta t}>\lim_{\Delta t\to -0}\frac{\chi(\rho+\Delta t)-\chi(\rho)}{\Delta t}\geqq 0.$$

The eqality at the right hand holds if and only if the body $K(\rho)$ is a cap-body of a sphere or a tangential body of a sphere.

In any case, these results (8_1), (8_2) and (8_3) express that the isoperimetric coefficient $\chi(t)$ is an increasing function all over the interval $0<t<\infty$. Moreover, the volume $V(t)$ and the surface area $S(t)$ are continuous in the same interval. Hence we have

THEOREM 1. *The isoperimetric coefficient $\chi(t)$ of the parallel sequence $\{K(t): 0\leqq t<\infty\}$ to a convex body K with inradius r (>0) is a monotone increasing* *function all over the interval* $0 < t < \infty$. *If the convex body* $K(r)$ *is a cap-body of a sphere, or a tangential body of a sphere, the isoperimetric coefficient is constant in the interval* $0 < t \leqq r$. *Further, if and only if the body* $K(r)$ *is a sphere, the isoperimetric coefficient is equal to unity all over the interval* $0 < t < \infty$.

4. The similitude-indicatrix sequence.

Let K be a convex body with inradius r (>0) and $\{K(t) : 0 \leqq t \leqq r\}$ the interior parallel sequence to the convex body K. At this time, the kernel $K(0)$ of the sequence is the set of the centers of the inspheres of $K(t)$, $0 < t \leqq r$ and is classified into three types, that is, *a point-kernel, line-kernel* and *plane-kernel* (see [4, P. 2]). In any case, taking a point of the kernel $K(0)$, we define it as the origin of the space. Let $K(t)$ be a member of the sequence $\{K(t) : 0 \leqq t \leqq r\}$, $\bar{A}(t)$ an *extreme supporting plane* of $K(t)$, a the normal indicatrix point of $\bar{A}(t)$ ant $H(a, t)$ the supporting function of the supporting plane $\bar{A}$(t). The supporting function $H(a, t)$ is dependent on the normal indicatrix point a and parameter t of the sequence We call the intersection $\bar{A}(t) \cap K(t)$ the *supporting set* of $K(t)$ and denote it by $A(t)$. Further let us denote the *relative inradius* of the supporting set $A(t)$ by $d(A(t))$. Then we have the following theorem ([4, P. 24]):

When the parameter t *of the sequence* $\{K(t) : 0 \leqq t \leqq r\}$ *decreases from* t *to* $t\text{-}d(A(t))$, *the supporting set* $A(t)$ *vanishes at the value* $t\text{-}d(A(t))$ *of parameter.*

Now let us denote by ρ_A the value $t\text{-}d(A(t))$ of parameter. Further if we denote by $H(a, \rho_A)$ the value $\lim\limits_{t \to \rho_A + 0} H(a, t)$, it follows by the definition of the interior parallel sequence that

$$H(a, t) - t = H(a, \rho_A) - \rho_A \geqq 0 .$$

The equality at the right hand holds only when $\rho_A = 0$, that is, *the supporting set* $A(t)$ *converges to the kernel* $K(0)$ *or a part of the kernel* $K(0)$. Further, if we denote the value $H(a, \rho_A) - \rho_A$ by $h(a)$ we have the following expression:

$$H(a, t) = t + h(a), \tag{9}$$

where $h(a)$ *is equal to zero when the supporting set* $A(t)$ *converges to the kernel or its part as* $t \to 0$.

At this time, taking the origin as the center of similitude, let us consider such *a set of all convex bodies which are similar and similarly situated to a convex body* $K(t)$ $(0 < t \leqq r)$. Further, taking a convex body which belongs to the set and whose inscribed sphere is a unit sphere, we denote it by $\Gamma(t)$ and call it the *similitude-indicatrix* of the set or the convex body $K(t)$. Then it follows by the property (**iii**) concerning with the isoperimetric coefficient $\chi(K)$ that

$$\chi(\Gamma(t)) = \chi(t).$$

That is to say, *the isoperimetric coefficient* $\chi(K)$ *of the convex body* K *is in* *general equal to the one* $\chi(\Gamma)$ *of the similitude-indicatrix* Γ.

On the other hand, it follows by the definition of the similitude that the similitude-indicatrix $\Gamma(t)$ has an *extreme supporting plane*, say $\bar{\alpha}(t)$, which is parallel to the corresponding extreme supporting plane $\bar{A}(t)$ of the convex body $K(t)$. Hence the corresponding two extreme supporting planes $\bar{A}(t)$ and $\bar{\alpha}(t)$ have the same normal indicatrix point a in common.

Hence, if we denote by $\eta(a, t)$ the supporting function of the similitude-indicatrix $\Gamma(t)$, $(0<t\leqq r)$ which corresponds to the one $H(a, t)$ *of* $K(t)$, we have by (9)

$$\eta(a, t) = 1 + \frac{h(a)}{t}, \qquad 0<t\leqq r. \tag{10}$$

Therefore, the supporting function $\eta(a, t)$ of the similitude-indicatrix $\Gamma(t)$ which corresponds to the same normal indicatrix point a is a monotone decreasing function with respect to parameter t. Or we have

$$\eta(a, t_i) > \eta(a, t_j), \qquad \rho_A \leqq t_i < t_j < r. \tag{11}$$

At this time, we have the following theorem ([1, P. 27]) :

A closed convex body is the intersection of the closed half spaces bounded by its extreme supporting planes.

Then we have by the theorem and the relation (11) that, if $0<t_1<t_2<\cdots<t_n<r$, it follows that

$$\Gamma(t_1) \supset \Gamma(t_2) \supset \cdots \supset \Gamma(r). \tag{12}$$

Further we have by the theorem 1 and the property **(iii)** with respect to the isoperimetric coefficient that

$$\chi(t_1) \leqq \chi(t_2) \leqq \cdots \leqq \chi(r). \tag{13}$$

Each of the equalities in (13) *holds when each of two convex bodies which are combined by the equality is a cap-body of a sphere or a tangential body of a sphere respectively.*

In the next place, defining the similitude-indicatrix sequence in $r<t<\infty$ in the same way as before and combining together both of them in $0\leqq t\leqq r$ and $r<t<\infty$, we define the *similitude-indicatrix sequence* of the parallel sequence $\{K(t) : 0\leqq t<\infty\}$ to a convex body K and denote it by $\{\Gamma(t) : 0\leqq t<\infty\}$.

At this time, it is easy to see that the form-figure $\Pi(t)$ and the normal indicatrix of the exterior parallel body $K(t)$ or its similitude-indicatrix $\Gamma(t)$ $(r<t<\infty)$ are always identical with the unit sphere itself whose center is at the origin. That is to say, all the supporting planes of the exterior parallel body $K(t)$ or the indicatrix $\Gamma(t)$ are the extreme ones all over the interval $r<t<\infty$. Here let us denote by $\Omega(O)$ the unit sphere with center at the origin O. The extreme supporting plane $\bar{A}(t)$ of the exterior parallel convex body

$K(t)$ $(r<t<\infty)$ can be defined all over the normal indicatrix $\Omega(O)$ of $K(t)$ and the supporting function $H(a,t)$ concerning to the supporting plane $\bar{A}(t)$ can be expressed as follows :

$$H(a,t)=t+h(a), \qquad h(a)\geqq 0,$$

where $h(a)$ is a constant and equal to zero when the supporting plane $\bar{A}(t)$ is a tangent plane to the insphere of $K(t)$. Further it is easy to see that, if the interior parallel body $K(t')$ $(0<t'\leqq r)$ has the extreme supporting plane $\bar{A}(t')$ which is parallel to the above supporting plane $\bar{A}(t)$, the constant $h(a)$ is common to their supporting functions $H(a,t')$ and $H(a,t)$.

In the next place, in the same manner as before, if we donote by $\eta(a,t)$ the supporting function of the similitude-indicatrix $\Gamma(t)$ in a direction a, it is expressed as follows :

$$\eta(a,t)=1+\frac{h(a)}{t} \qquad r<t<\infty. \tag{14}$$

Hence the *supporting function* $\eta(a,t)$ *in a direction* a *is a monotone decreasing function with respect to the parameter* t. If the parameter tends to infinity, the supporting function $\eta(a,\mathrm{t})$ of the indicatrix $\Gamma(\mathrm{t})$ converges to unity. Moreover it holds for all the directions which are defined on the unit sphere $\Omega(O)$. Therefore we can say that when the parameter t tends to infinity, the similitude-indicatrix $\Gamma(t)$ converges to the unit sphere $\Omega(O)$. Or we have

$$\lim_{t\to+\infty}\Gamma(t)=\Omega(O). \tag{15}$$

Further, it is evident that *no matter how any other convex body may be taken instead of the original convex body* $K(r)$, *the limiting figure in the formula* (15) *is always equal to the unit sphere* $\Omega(O)$.

Thus we have

THEOREM 2. *Let* K *be a convex body,* $\{\Gamma(t):0\leqq t<\infty\}$ *the similitude-indicatrix sequence which is defined referring to* K *and* $\Omega(O)$ *a unit sphere with center at the origin* O. *Then the superior limiting figure of the similitude-indicatrix sequence is always the unit sphere* $\Omega(O)$.

The result is considered to be the one corresponding to *Steiner's Kugelungs-theorem*[5] which is obtained by means of symmetrization. At this time, *Steiner's symmetrization leaves the volume unchanged while our transformation which corresponds the body* $K(t)$ *with the similitude-indicatrix* $\Gamma(t)$ *leaves the shape unchanged.* Further we may add

COROLLARY. *The superior limit of the isoperimetric coefficient of the similitude-indicatrix sequence is always equal to unity.*

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5) See [2, PP. 26–28].

On the other hand, according to the monotone decreasing property of the supporting function $\eta(a, t)$ of the similitude-indicatrix $\Gamma(t)$, we can express the formula (15) as follows:

If $r < t_1' < t_2' < \cdots\cdots$, *it holds that*

$$\Gamma(r) \supset \Gamma(t_1') \supset \Gamma(t_2') \supset \cdots\cdots \supset \Omega(0) \tag{16}$$

and

$$\chi(r) \leqq \chi(t_1') \leqq \chi(t_2') \leqq \cdots\cdots \leqq 1. \tag{17}$$

Since the form-figure $\Pi(t)$ in the present case $(r < t < \infty)$ is always congruent to the unit sphere $\Omega(O)$, by using the condition for the equality in the formula (8_1), we can say as follows: *the necessary and sufficient condition for the equality in* (17) *is that the similitude-indicatrix* $\Gamma(t)$ *is a sphere.* That is to say, *all the equalities in* (17) *hold simultaneously, if the convex body* $K(r)$ *is a sphere, and vice versa.*

Further by using the monotone increasing property of the isoperimetric coefficient $\chi(t)$ and the theorem 2 we have

THEOREM 3. *Of all convex bodies in the three dimensional Euclidean space* E_3, *the sphere has the greatest isoperimetric coefficient.*

On the other hand, we denote by $b(u, t)$ the *breadth* of the similitude-indicatrix $\Gamma(t)$ in the direction u. Further, if we denote by $D(t)$ and $\Delta(t)$ the *diameter* and the *width*[6] *of the body* $\Gamma(t)$ respectively, it is easy to prove by (12) and (16) that:

The diameter $D(t)$ *and the width* $\Delta(t)$ *of the similitude indicatrix* $\Gamma(t)$ *are monotone decreasing functions of parameter* t *in* $0 < t < \infty$.

Further let us define the *amplitude* of the similitude-indicatrix $\Gamma(t)$ as follows:

DEFINITION. *The difference* $D(t)-\Delta(t)$ *of the diameter* $D(t)$ *and width* $\Delta(t)$ *of the similitude-indicatrix* $\Gamma(t)$ *is said the amplitude of the body* $\Gamma(t)$ *or the relative amplitude of the parallel sequence* $\{K(t) : 0 \leqq t < \infty\}$ *and denote it by* $\sigma(t)$.

Now, taking the the sequence $\{\Gamma(t) : r < t < \infty\}$, let us examine the change of the value $\sigma(t)$ in $r < t < \infty$. Since the values $D(t)$ and $\Delta(t)$ should be realized by the breadths $b(u, t)$ of the body $\Gamma(t)$ respectively, let us suppose that the breadths $b(u, t)$ and $b(v, t)$ in the directions u and v are equal to the diameter $D(t)$ and the width $\Delta(t)$ of $\Gamma(t)$ respectively. At this time, we have $b(u, t) = b(-u, t)$ and $b(v, t) = b(-v, t)$.

Then applying the formula (14), we can express $D(t)$ and $\Delta(t)$ as follows:

$$D(t) = b(u, t) = \eta(u, t) + \eta(-u, t) = 2 + \frac{h(u) + h(-u)}{t},$$

$$\Delta(t) = b(v, t) = 2 + \frac{h(v) + h(-v)}{t}.$$

Here since the constant $h(a)$ in (14) is finite, we can say that:

6) See the definitions in [2, P. 10].

THEOREM 4. *The relative amplitude $\sigma(t)$ of the parallel sequence $\{K(t): 0 \leqq t < \infty\}$ to a convex body K is a monotone decreasing function with respect to the parameter in the interval $r < t < \infty$ and converges to zero as $t \to +\infty$.*

5. An isoperimetric sequence.

Let K be a convex body in the three dimensional Euclidean space E_3, S_0 the surface area $S(K)$ of the convex body K and $0 < S_0 < \infty$. Moreover, first let us suppose that the convex body K has a plane-kernel, that is, the point-set of all the centers of inspheres of K is a convex closed domain with interior points in a plane E. Then the plane-kernel is bounded and have a finite plane area, say F_0, $0 < F_0 < \infty$.

Now we define the interior parallel sequence $\{K(t): 0 \leqq t \leqq r\}$ of the convex body K. Then we have $\lim_{t \to 0} S(t) = 2F_0$. Further, let us suppose that we obtain the parallel sequence $\{K(t): 0 \leqq t < \infty\}$ to the convex body K. First, taking the parallel sequence $\{K(t): 0 < t < \infty\}$, let us define the similitude-indicatrix sequence $\{\Gamma(t): 0 < t < \infty\}$ in the same open interval $0 < t < \infty$. At this time, let us examine the inferior limiting figure $\lim_{t \to 0} \Gamma(t)$ of such a similitude-indicatrix sequence $\{\Gamma(t): 0 < t < \infty\}$. For the purpose, taking the plane-kernel $K(0)$ of the interior parallel sequence $\{K(\mathrm{t}): 0 \leqq t \leqq r\}$ and the plane, say E which contains the kernel $K(0)$, let us choose two points O and P such that O is an interior point of the kernel $K(0)$ and P a frontier point of the plane-kernel $K(0)$ in the plane E. We define the point O as the origin of the space and the center of similitude. Further, let us denote by $\mathfrak{S}(P, t)$ the sphere whose center is P and whose radius is equal to t, $0 < t \leqq r$. Then it is easy to see that if an extreme supporting plane $\bar{A}(t)$ of $K(t)$ is parallel to the plane kernel $K(0)$, the supporting set $A(t)$ converges to the kernel $K(0)$ as $t \to 0$ and if an extreme supporting plane $\bar{A}(t)$ of $K(t)$ is not parallel to the plane-kernel $K(0)$ and is in common to $K(t)$ and $\mathfrak{S}(P, t)$, the corresponding extreme supporting set $A(t)$ converges to the center P of the sphere $\mathfrak{S}(P, t)$ as $t \to 0$.

Then the supporting function $H(a, t)$ of an extreme supporting plane $\bar{A}(t)$ which is common to $\mathfrak{S}(P, t)$ and $K(t)$ is expressed as follows:

$$H(a, t) = t + h(a), \qquad 0 \leqq h(a) < \infty$$

where $h(a)$ is a constant in the interval $0 \leqq t < \infty$ and it is equal to zero when the corresponding supporting plane is parallel to the plane kernel $K(0)$. Then the supporting function $\eta(a, t)$ of the image supporting plane $\bar{\alpha}(t)$ of the similitude indicatrix $\Gamma(t)$ $(0 < t \leqq r)$ is expressed as follows:

$$\eta(a, t) = 1 + \frac{h(a)}{t}, \qquad 0 \leqq h(a) < \infty$$

where the equality at the right hand holds when $\bar{A}(t)$ is parallel to the plane-kernel

$K(0)$. Therefore, in case of such an extreme supporting plane as it is not parallel to the plane-kernel $K(0)$ we have

$$\lim_{t\to+0} \eta(a, t) = \infty.$$

On the other hand, in case of two extreme supporting planes which are parallel to the plane-kernel $K(0)$, the corresponding supporting functions $\eta(a, t)$ are constantly equal to unity, that is,

$$\eta(a, t) = 1, \qquad 0 \leqq t \leqq r.$$

Hence the limiting figure $\lim_{t\to+0} \Gamma(t)$ is an unbounded plate of breadth 2 which is bounded by two supporting planes which are parallel to the plane-kernel $K(0)$. On the other hand, we have seen in §4 that the superior limiting figure $\lim_{t\to+\infty} \Gamma(t)$ is a unit sphere.

Lastly, taking up all the member of the so-obtained similitude-indicatrix sequence $\{\Gamma(t) : 0 < t < \infty\}$, let us define such a set of similar transformations to them as each member of the transformed sequence by the transformation has always the same surface area S_0 and denote it by $\{\mathfrak{K}(t) : 0 < t < \infty\}$. Further, let us add the extreme member which corresponds to the value 0 of parameter. That is to say, we define the extreme figure $\mathfrak{K}(0)$ of the sequence to be a convex figure in the plane E which is similar and similary situated to the plane-kernel $K(0)$ and whose plane area is equal to $S_0/2$. On the other hand, let us denote by $\mathfrak{K}(\infty)$ the sphere with surface area S_0. Then it is clear that, *if* $t \to +\infty$, $\mathfrak{K}(t)$ *converges to the sphere* $\mathfrak{K}(\infty)$. Thus the isoperimetric sequence has been obtained including both of the extreme figures. By our definition it is clear that $\mathfrak{K}(r)$ is congruent to the original convex body $K(r)$. Moreover since the original body $K(r)$ has the plane-kernel, the isoperimetric coefficient $\chi(\Gamma(t))$ of the similitude-indicatrix sequence $\{\Gamma(t) : 0 < t < \infty\}$ is a strictly monotone increasing function. Thus we have

$$\chi(\Gamma(t_1)) < \chi((\Gamma(t_2)), \qquad 0 < t_1 < t_2.$$

Or it follows by the similitude that

$$\chi(\Gamma(t_1)) = \chi(\mathfrak{K}(t_1)) \quad \text{and} \quad \chi(\Gamma(t_2)) = \chi(\mathfrak{K}(t_2)).$$

Hence we have

$$(19) \qquad \chi(\mathfrak{K}(t_1)) < \chi(\mathfrak{K}(t_2)), \qquad 0 < t_1 < t_2.$$

On the other hand, by the definition of the sequence $\{\mathfrak{K}(t) : 0 < t < \infty\}$ we have

$$(20) \qquad S(\mathfrak{K}(t_1)) = S(\mathfrak{K}(t_2)) = S_0, \qquad 0 < t_1 < t_2.$$

Hence we have by (19), (20) and the definition of the isoperimetric coefficient

$$V(\mathfrak{K}(t_1)) < V(\mathfrak{K}(t_2)), \qquad 0 < t_1 < t_2. \tag{21}$$

Thus we have shown *an example of the isoperimetric sequence in the case when the extreme figure $\mathfrak{K}(0)$ is a two dimensional convex figure of plane area $S_0/2$.* Moreover we can give *an example of the isoperimetric sequence by the same procedure in the case when $\mathfrak{K}(0)$ is a straight line.*

Now, let us define

DEFINITION. *The amplitude $\sigma(t)$ of the similitude-indicatrix $\Gamma(t)$ is called the relative amplitude of a member $\mathfrak{K}(t)$ of the isoperimetric sequence $\{\mathfrak{K}(t): 0<t<\infty\}$.*

The relative amplitude $\sigma(t)$ of a member $\mathfrak{K}(t)$ of the sequence $\{\mathfrak{K}(t): 0<t<\infty\}$ indicates the degree of the convergence of $\mathfrak{K}(t)$ which tends to the superior limiting sphere. At this time, it follows by the theorem 4 that relative amplitude $\sigma(t)$ of $\mathfrak{K}(t)$ is a strictly monotone decreasing function in the open interval $r<t<\infty$. Now, collecting this result and the inequality (21), we have

THEOREM 5. *The relative amplitude $\sigma(t)$ of a member $\mathfrak{K}(t)$ of the isoperimetric sequence $\{\mathfrak{K}(t): 0<t<\infty\}$ with a given surface area S_0 $(0<S_0<\infty)$ is a strictly monotone decreasing function in $r<t<\infty$ and converges to zero as $t\to\infty$. The volume $V\{\mathfrak{K}(t)\}$ of the member $\mathfrak{K}(t)$ is a strictly monotone increasing function in $0<t<\infty$ and $\mathfrak{K}(t)$ converges to the sphere of radius $\sqrt{S_0/4\pi}$ as $t\to+\infty$.*

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