The Science Reports of the Kanazawa University, Vol. VII, No. 2, pp.41—54, August 1961.

On the sufficient conditions for some boundary component of a domain bounded by an infinite number of circles to be parabolic

By

Tohru AKAZA*

(Received January 27, 1961)

1. Let D be a domain in the complex z-plane and γ be a boundary component of D consisting of a single point. The component γ is said to be weak if its image under any conforml mapping of D consists of a single point. If γ is not weak, then we say that γ is unstable (Sario [14]).

Consider circles $c_\nu(\nu=\pm 1, \pm 2, \cdots)$ with centers ξ_ν on the real axis of the z-plane such that they are disjoint from each other and cluster to infinity $z=\infty$ from the both sides of the real axis. Here, without loss of generality, we may assume that $\xi_{-\nu-1}<\xi_{-\nu}<0<\xi_\nu<\xi_{\nu+1}$ for every positive integer ν. Let B the fundamental domain, bounded by $c_\nu(\nu=\pm 1, \pm 2, \cdots)$, of a properly discontinuous group generated by the hyperbolic linear transformations with real coefficients

$$z'=S(z)=\frac{\alpha_\nu z+\beta_\nu}{\gamma_\nu z+\delta_\nu}, \qquad (\nu=\pm 1, \pm 2, \cdots), \tag{1.1}$$

each of which for every ν transforms the outside of $c_{-\nu}$ into the inside of c_ν.

The purpose of this paper is to investigate the relation between the weakness of the boundary component ∞ of B and the parabolic type of it defined by the function which is generated from Γ ([4], [11]), and to get the sufficient conditions for it to be parabolic.

2. Consider the Poincaré theta-series of (−2)-dimension

$$\textcircled{H}(z)=\sum_{\Gamma} H[S(z)]\frac{dS(z)}{dz}, \tag{2.1}$$

where the kernelfunction $H(z)$ is a real rational function whose poles are in the set $\overline{B}=B\cup\left(\bigcup_{\nu=-\infty}^{\infty} c_\nu\right)$. It is well known that the series (2.1) converges absolutely and uniformly in the complement D^* of the set of singular points of Γ, with respect to the z-plane, and defines a function meromorphic in D^*. For each transformation of Γ, we have the well known differential invariant

$$\textcircled{H}(S(z))dS(z)=\textcircled{H}(z)dz. \tag{2.2}$$

This invariant is called an automorphic differential. The function

* Department of Mathematics, Kanazawa University

$$(2.3) \qquad I(z)=\int_{z_0}^{z} \textcircled{H}(z)dz$$

is obtained by integrating the automorphic differential along an arbitrary path in D^*. For any $S(z)\in\Gamma$, the following relation

$$(2.4) \qquad I(S(z))=I(z)+k_s$$

is satisfied, where k_s is the additive constant which depends on $S(z)$.

Now, if we choose as a kernelfunction

$$(2.5) \qquad H(z)=\frac{1}{z-a}-\frac{1}{z-b}, \qquad (a<b,\ \text{real},\ a,\ b\in\overline{B}),$$

then we obtain the following analytic representation of $I(z)$:

$$(2.6) \quad I(z)=\sum_{\Gamma}\log\left[\frac{S(z)-a}{S(z)-b}:\frac{S(z)-a}{S(z)-b}\right]=\sum_{\Gamma}\log\left[\frac{z-S(a)}{z-S(b)}:\frac{z_0-(a)}{z_0-(b)}\right].$$

In what follows, we assume that z_0 is the origin $z=0$ for convenience.

The following two cases (i) and (ii) occur according to the positions of a and b.

(i) The case where a and b are congruent with respect to some generator of Γ, that is, $b=S_\nu(a)$ for ν. In this case, the poles of different terms of (2.6) are canceled each other in pairs and we obtain a finite integral in D^*. Moreover, we can easily see that $I(z)$ depends on the pole $J_\nu=-\frac{\delta_\nu}{\gamma_\nu}$ of $S_\nu(z)$ but does not depend on a. If we denote such an $I(z)$ by

$$(2.7) \qquad \varphi_\nu(z)=\int \textcircled{H}(z,\ J_\nu)\ dz,$$

then we have a squence of functions $\{\varphi_\nu(z)\}$ $(\nu=1,\ 2,\ 3,\cdots)$. If $\xi_\nu=-\xi_{-\nu}$ and if the radius of c_ν equals that of $c_{-\nu}$, then the function $\varphi_\nu(z)$ is a real elementary normal integral of the first kind in the sense of L. Myrberg [6]. We call $\varphi_\nu(z)$ a real normal integral of the first kind.

By an easy computation (Burnside [4], P. J. Myrberg [9, 11]) we obtain the relations

$$(2.8) \qquad \int_{c_\nu} d\varphi_\nu=2\pi i, \qquad \int_{c_\mu} d\varphi_\nu=0 \qquad (\mu\neq\nu).$$

If γ_ν is a Jordan curve which joins two equivalent points on circles $c_{-\nu}$ and c_ν in the upper half of B, then the period

$$(2.9) \qquad \tau_{\nu\mu}=\int_{\gamma_\mu} d\varphi_\nu$$

of φ_ν along γ_μ is real. By a simple calculation it holds the symmetric relation

$$(2.10) \qquad \tau_{\mu\nu}=\tau_{\nu\mu}.$$

(ii) The case where a and b are not congruent for any generator of Γ. The poles of different terms of (4) cannot be canceled each other. We denote such an integral $I(z)$ by $\chi_{ab}(z)$ and call it a real normal integral of the third kind (P. J. Myrberg [10, 11, 12]). It has the following properties :

1° $\chi_{ab}(z)$ is regular in B except at a and b, where it has logarithmic poles with residues -1 and 1 repectively.

2° The periods of $\chi_{ab}(z)$ along c_ν and γ_ν are

$$\int_{c_\nu} d\,\chi_{ab}(z)=0 \qquad (\nu=\pm 1,\ \pm 2,\cdots), \tag{2.11}$$

and

$$\int_{\gamma_\nu} d\,\chi_{ab}(z)=\varphi_\nu(b)-\varphi_\nu(a) \qquad (\nu=1,\ 2,\cdots). \tag{2.12}$$

3. Let B_0 be the upper half of the fundamental domain B. Any branches of $\varphi_\nu(z)$ and $\chi_{ab}(z)$ are single-valued and regular in B_0 by the monodrnmy theorem. We take the branches of $\varphi_\nu(z)$ and $\chi_{ab}(z)$ such that $\varphi_\nu(0)=0$ and $\chi_{ab}(0)=0$ and denote them by $\varphi_\nu(z)$ and $\chi_{ab}(z)$ again. Let us consider the images of B_0 by them. The function $\varphi_\nu(z)$ is real on the intersection of $\overline{B}$ with the part of the real axis between $c_{-\nu}$ and c_ν. The imaginary part of $\varphi_\nu(z)$ increases by π, when z describes the upper half circumference of c_ν or $c_{-\nu}$. According as the origin $z=0$ is contained in the interval $[a, b]$ or not, $\chi_{ab}(z)$ is real on the real axis in B inside or outside $[a, b]$. The imaginary part of $\chi_{ab}(z)$ increases by $-\pi$ or π in the former and by π or $-\pi$ in the latter respectively, when z passes through $z=a$ or $z=b$ in the positive direction.

We see that $w=\varphi_\nu(z)=u_\nu(z)+iv_\nu(z)$ maps B_0 conformally onto the rectangle $a_\nu< u_\nu < a_\nu+\tau_{\nu\nu}$, $0< v_\nu <\pi$ with vertical slits starting from the upper and lower sides and corresponding to the upper halves of all c_μ except for $\mu=\nu$. And $w=\chi_{ab}(z)=u_{ab}(z)+iv_{ab}(z)$ maps B_0 conformally onto the strip domain $-\infty< u_{ab} <\infty$, $0< v_{ab} <\pi$ with vertical slits starting from the upper and the lower sides and corresponding to the upper halves of $c_\mu(\mu=\pm 1,\ \pm 2,\cdots)$. (Fig. (a), (b))

As to these slits, there are two cases : these slits cluster to a point from the both sides or not. In the former case we say that the type of $\varphi_\nu(z)$ or $\chi_{ab}(z)$ is parabolic and in latter case we say non-parabolic.

Then we obtained the following results ([1]) :

Theorem 1. *If $\chi_{ab}(z)$ is parabolic with respect to some* (a, b) $(-\infty< a< b<\infty)$, *then $\chi_{ab}(z)$ is also parabolic with respect to any pair* (a, b).

In the case where a and b are congruent with respect to some generator $S_\nu(z)\in\Gamma$, we can prove the following

Theorem 2. *Whether the type of $\varphi_\nu(z)$ is parabolic or not is independent of ν ; more precisely $\varphi_\nu(z)$,* $(\nu=1,\ 2,\cdots)$ *are all parabolic or all non--parabolic.*

From the above theorems we find that the type of $\varphi_\nu(z)$ or $\chi_{ab}(z)$ depends on the behavior of circles in the neighborhood of ∞, that is, the concept of the type of $\varphi_\nu(z)$ or $\chi_{ab}(z)$ is a local property. Hereafter we say that the component ∞ is parabolic or non-parabolic according as the type of $\varphi_\nu(z)$ or $\chi_{ab}(z)$ is parabolic or non-parabolic.

Considering the images of B by $e^{\varphi_\nu(z)}$ and $e^{\chi_{ab}(z)}$, we find that they are a ring domain and a plane slitted along the concentric circular arcs, whose centers are at the origin and which are symmetric with respect to the real axis, since B_0 and B_1 are symmetric with respect to the real axis. (Fig. (c) (d)).

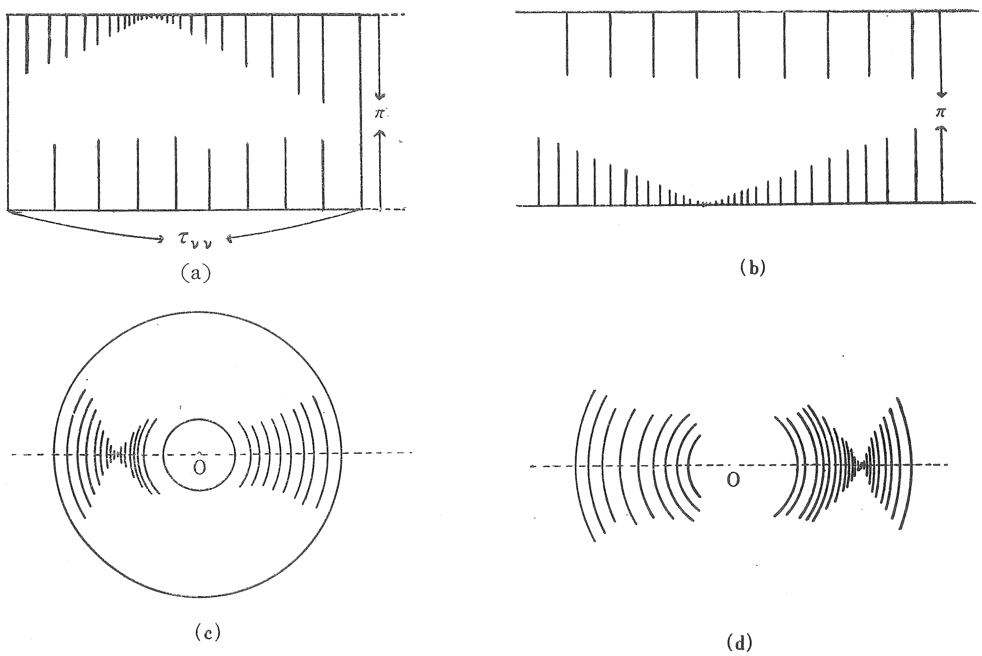

Fig.

4. Let us consider the relation between the weakness and the parabolic type of the component ∞.

By a canonical conformal mapping, B can be mapped onto a plane domain slitted along an infinite number of concentric circular arcs symmetric with respect to the real axis whose common centers are the origin, where the component ∞ corresponds to the origin.

Sario ([14]) proved that the necessary and sufficient condition that the component ∞ be weak is that the circular slits converge to the origin.

But in our case B can be mapped by $e^{\varphi_\nu(z)}$ and $e^{\chi_{ab}(z)}$ onto a ring domain and a plane domain slitted along an infinite number of concentric circular arcs with common centers at the origin symmetric with respect to the real axis, which converge to the real point different from the origin. (Fig. (c), (d)).

Then the following question arises : Does the weakness equal the parabolic type ?

Since the parabolic type is a local property, we may treat the problem in the neighborhood of the accumulating point of slits of the rectangle and the strip domain with slits starting from the upper and lower sides, which is the image of B by $w=\varphi_\nu(z)$ and $\chi_{ab}(z)$. By a suitable linear transformation, the accumulating point of slits is carried into the origin. Then we may investigate the parabolic type and the weakness in the neighborhood of the origin in the plane slitted along slits being symmetric and orthogonal to the real axis and converging to the origin from the both sides.

In my former paper ([2]) we treated such problem.

Let $S_n (n=1, 2, \cdots)$ be a sequence of slits being symmetric and orthogonal to the positive real axis of the complex w-plane and converging to the origin $O: w=0$. We delete the set $\bigcup_{n=1}^{\infty} S_n \cup \{O\}$ from the w-plane and denote by D the resulting domain. Then we proved the following:

Theorem 3. *If* $S_n (n=1, 2, \cdots)$ *are segments*: $x=a_n (>0)$, $|y| \leqq h_n$ *satisfying* $0 < a_{n+1} < a_n$, $\lim_{n\to\infty} a_n = 0$ *and*

$$h_n \leqq a_n \tan \alpha = h_n' \tag{4.1}$$

for some fixed $\alpha \left(0 < \alpha < \frac{\pi}{2}\right)$, *then* O *is a weak boundary component of the domain by deleting* $\bigcup_{n=1}^{\infty} S_n \cup \{O\}$ *from the w-plane.*

Moreover we showed that in the case when segments in our Theorem 3 do not satisfy the condition (4.1) the origin O is not always weak.

Theorem 4. *If* $S_n (n=1, 2, \cdots)$ *are segments*: $x=\frac{1}{n}$, $|y| \leqq c\left(\frac{1}{n-1}\right)^p$, $(c>0,\ 0<p<1)$, *then the origin is an unstable boundary component of the domain obtained by deleting* $\bigcup_{n=1}^{\infty} S_n \cup \{O\}$ *from the w-plane.*

Recently the extensions of the above theorems were obtained by the author and *K.* Oikawa ([3]).

Our problem is solved completely. Because we may divide the real axis into the positive and negative ones and use the above theorems. Therefore the parabolic type is different from the weakness and is divided into the weak and the unstable parts.

5. Now we want to get the sufficient condition for the componet ∞ of B to be parabolic. From Theorems 1 and 2 it is enough to decide the parabolic type of $\varphi_\nu(z)$.

In difference with the symmetric group, asymmetry of the arrangement of circles $\{c_\nu\}$ with respect to the imaginary axis becomes a subject of discussion.

Let us denote the parts of the real axis between the $c_{\pm\nu}$ and $c_{\pm(\nu+1)}$ by $k_{\pm\nu}$, of which endpoints are $e_{\pm2\nu}$ and $e_{\pm(2\nu+1)}$. Without loss of generality we may assume that the radii $R_{\pm\nu}$ of circles $c_{\pm\nu}$ are bounded, that is, $R_{\pm\nu} \leqq 1$. We describe the circles $c'_{\pm\nu}$ whose diameters are the intervals $k_{\pm\nu}$ between $c_{\pm\nu}$ and $c_{\pm(\nu+1)}$. Suppose that the radii $R'_{\pm\nu}$ of $c'_{\pm\nu}$ are bounded, that is, $R'_{\pm\nu} \leqq 1$. The hyperbolic linear transformation $T_\nu(z)$ which transforms $c'_{-\nu}$ to c'_ν is decided.

We describe the circle orthogonal to the real axis

$$H_t^{(\nu)};\ |z-a_\nu(t)| = \rho_\nu(t),\ (0 \leqq t \leqq 1), \tag{5.1}$$

which go through a point $p'_{-\nu}$ on $c'_{-\nu}$ and the corresponding point p'_ν on c'_ν by $T_\nu(z)$. This circle $H_t^{(\nu)}$ intersect the real axis at the points $e_{-\nu}(t)$ and $e_\nu(t)$, where the centers $H_0^{(\nu)}$ and $H_1^{(\nu)}$ are

$$(5.2) \qquad a_\nu(0)=\frac{e_{2\nu}+e_{-2\nu}}{2}, \qquad a_\nu(1)=\frac{e_{2\nu+1}+e_{-(2\nu+1)}}{2}.$$

We can easily see that

$$(5.3) \qquad \rho_\nu(t)<\rho_\nu(t'),\ (t<t');\ \rho_\nu(t)<\rho_{\nu'}(t),\ (\nu<\nu'); \quad \rho_\nu(t)\to\infty,\quad (\nu\to\infty).$$

Let

$$(5.4) \qquad L(\rho_\nu(t))=\int_{H_t^{(\nu)}}|d\,\varphi_n|$$

be the length of the image of the upper half circle of (5.1) by $\varphi_n(z)$. This is the length of the curve which joins two points on the horizontal side of the rectangle that is the image of B_0 by $\varphi_n(z)$. If we denote a point on $H_t^{(\nu)}$ and the derivative $a_\nu(t)$ with respect to $\rho_\nu(t)$ by

$$(5.5) \qquad z=a_\nu(t)+\rho_\nu(t)e^{i\alpha},\ \frac{d\,a_\nu(t)}{d\,\rho_\nu(t)}=a_\nu'(t)$$

respectively, we obtain by Schwarz's inequality

$$(5.6) \qquad \{L(\rho_\nu(t))\}^2=\left(\int_{H_t^{(\nu)}}|\varphi_n'(z)|\,|dz|\right)^2\leqq\int_{H_t^{(\nu)}}|\varphi'_n(z)|^2|1-a_\nu'(t)\cos\alpha|\,|dz| \cdot\int_{H_t^{(\nu)}}\frac{|dz|}{|1-a_\nu'(t)\cos\alpha|}$$

Since $|a_\nu'(t)|<1$, it holds

$$(5.7) \qquad \frac{\{L(\rho_\nu(t))\}^2}{\pi}\ \frac{1-|a_\nu'(t)|}{\rho_\nu(t)}<\int_{H_t^{(\nu)}}|\varphi_n'(z)|^2|1-a_\nu'(t)\cos\alpha|\,|dz|.$$

Suppose that

$$(5.8) \qquad L(\rho_\nu(t))\geqq\varepsilon>0.$$

Multiplying $d\rho_\nu(t)$ in both sides of (5.7) and integrating from $\rho_\nu(0)$ to $\rho_\nu(1)$, then we have the following inequality,

$$(5.9) \qquad \frac{\varepsilon^2}{\pi}\int_{\rho_\nu(0)}^{\rho_\nu(1)}\frac{1-|a_\nu'(t)|}{\rho_\nu(t)}\,d\rho_\nu(t)<\int_{B(e_{2\nu},e_{2\nu+1})}|\varphi_n'(z)|^2\ dA=D_{B(e_{2\nu},\ e_{2\nu+1})}(\varphi_n),$$

where $B(e_{2\nu},\ e_{2\nu+1})$ is the part bounded by $H_0^{(\nu)}$, $H_1^{(\nu)}$ and the real axis, and $dA=|1-a_\nu'(t)\cos\alpha|\,|d\alpha|\,|d\rho_\nu(t)|$ is its surface element and $D_{B(e_{2\nu},\ e_{2\nu+1})}(\varphi_n)$ is the area of the image.

Summing up from 1 to N with respect to ν we obtain

$$(5.10) \qquad \frac{\varepsilon^2}{\pi}\sum_{\nu=1}^{N}\int_{\rho_\nu(0)}^{\rho_\nu(1)}\frac{1-|a_\nu'(t)|}{\rho_\nu(t)}\,d\rho_\nu(t)<\sum_{\nu=1}^{N}D_{B(e_{2\nu},\ e_{2\nu+1})}(\varphi_n)\leqq D_{B_0}(\varphi_n)=\pi\,\tau_{nn}.$$

The right side of (5.10) is finite for fixed n, since $D_{B_0}(\varphi_n)$ is the area of the rectangle. If, under (5.8),

$$(5.11)\qquad \sum_{\nu=1}^{\infty}\int_{\rho_\nu(0)}^{\rho_\nu(1)}\frac{1-|a_\nu'(t)|}{\rho_\nu(t)}\,d\rho_\nu(t)=\infty,$$

it contradicts with the fact that $D_{B_0}(\varphi_n)$ is finite. Therefore if we suppose (5.11), then there exists ν_0 for any small ε, so that $L(\rho_\nu(t))<\varepsilon$ for any $\nu\geqq\nu_0$.

Then we have the following

Theorem 5. *If* (5.11) *establishes, then the boundary component* ∞ *is parabolic.*

Now let us consider the property of $a_\nu'(t)$ $(0\leqq|a_\nu'(t)|<1)$ in (5.10) which express the variation of the center caused by the variation of the radius of $H_t^{(\nu)}$ and provides the asymmetry of circles $\{c_{\pm\nu}\}$, $(\nu=1,2,\cdots)$ with respect to the imaginary axis.

Denote by $\theta_\nu(t)$ the function

$$(5.12)\qquad \frac{1}{1-|a_\nu'(t)|},\qquad (0\leqq t\leqq 1).$$

P. J. Myrberg inverstigated the property of $\theta_\nu(t)$ in his paper ([8]) about the type problem of a simply connected open Riemann surface. If we denote by m_ν the smaller distance from the pole of $T_\nu(z)$ to $e_{-2\nu}$ and $e_{-(2\nu+1)}$, we have

$$(5.13)\qquad \theta_\nu(t)<c\psi(\nu),$$

where c is a constant and $\psi(\nu)$ is

$$(5.14)\qquad \psi(\nu)=\max\left(\frac{R_\nu'}{R_{-\nu}'},\ \frac{R_{-\nu}'}{R_\nu'}\right)\cdot\left(\frac{2R_\nu'-m_\nu}{m_\nu}\right)^2,$$

which is independent of t. If we suppose

$$(5.15)\qquad \psi(\nu)<\log\rho_\nu(t),$$

we obtain from the left side of (5.10)

$$(5.16)\qquad \sum_{\nu=1}^{\infty}\int_{\rho_\nu(0)}^{\rho_\nu(1)}\frac{d\,\rho_\nu(t)}{\rho_\nu(t)\log\rho_\nu(t)}<c\cdot\sum_{\nu=1}^{\infty}\int_{\rho_\nu(0)}^{\rho_\nu(1)}\frac{1-|a_\nu'(t)|}{\rho_\nu(t)}\,d\rho_\nu(t).$$

On the other hand the left side is

$$(5.17)\qquad \sum_{\nu=1}^{\infty}\int_{\rho_\nu(0)}^{\rho_\nu(1)}\frac{d\rho_\nu(t)}{\rho_\nu(t)\log\rho_\nu(t)}=\sum_{\nu=1}^{\infty}\Big[\log\log\rho_\nu(t)\Big]_{\rho_\nu(0)}^{\rho_\nu(1)}=\log\prod_{\nu=1}^{\infty}\frac{\log\rho_\nu(1)}{\log\rho_\nu(0)}$$

$$=\log\prod_{\nu=1}^{\infty}\left(1+\frac{\log\rho_\nu(1)-\log\rho_\nu(0)}{\log\rho_\nu(0)}\right).$$

If

$$(5.18)\qquad \sum_{\nu=1}^{\infty}\frac{\log\rho_\nu(1)-\log\rho_\nu(0)}{\log\rho_\nu(0)}$$

diverges, then (5.17) diverges, hence the right member of (5.16) diverges. Therefore we have the following

Theorem 6. *If the series* (5.18) *diverges under* (5.15), *then the component* ∞ *is parabolic.*

Especially if we suppose

$$0 \leqq |a_\nu'(t)| \leqq k < 1, \tag{5.19}$$

we get

$$\theta_\nu(t) \leqq \frac{1}{1-k} = K. \tag{5.20}$$

Hence we obtain the following

Corollary 1. *If under* (5.19) *the series*

$$\sum_{\nu=1}^{\infty} \frac{\rho_\nu(1)-\rho_\nu(0)}{\rho_\nu(0)} \tag{5.21}$$

diverges, the infinity is the parabolic boundary component.

If Γ is a symmetric group, that is, $k=0$, circles $\{c_\nu\}$ are symmetric with respect to the imaginary axis. The series (5.21) is modified and we obtain L. Myrberg's theorem ([6]) about the type problem of the real hyperelliptic integral of the first kind as the special case.

Corollary 2. (L. Myrberg) *In the case that* Γ *is a symmetric group, if the series*

$$\sum_{\nu=1}^{\infty} \frac{e_{2\nu+1} - e_{2\nu}}{e_{2\nu}} \tag{5.21}$$

divereges, the component ∞ *is parabolic.*

6. *Remark.* Savage ([15]) got many criteria for boundary component to be weak. Let us consider the relation between L. Myrberg's criterion and Savage's relative width criterion.

Consider a doubly connected bounded domain G of the complex plane. Let γ_1 and γ_2 denote its boundary curves. Let d be the distance between the closed sets γ_1 and γ_2. Consider all rectifiable curves c in G which separate γ_1 from γ_2, and which are at a distance $\geqq d/2$ from $\gamma_1 \cup \gamma_2$. Let L be the greatest lower bound of the lengths of these c. We define the relative width ω of G as $\omega = d/L$.

In the case that Γ is a symmetric group, we denote by $\{G_\nu\}$ a sequence of ring domains converging to ∞, where the boundary curves $\gamma_{2\nu}$ and $\gamma_{2\nu+1}$ of G_ν are circles whose common centers are the origin and whose radii are $R_{2\nu}=e_{2\nu}$ and $R_{2\nu+1}=e_{2\nu+1}$.

Then the relative width ω_ν of G_ν is $(e_{2\nu+1}-e_{2\nu})/\pi(e_{2\nu+1}+e_{2\nu})$. By Savage's criterion, if $\sum_{\nu=1}^{\infty} \omega_\nu$ is divergent, then the component ∞ is weak. Since $\sum_{\nu=1}^{\infty} (e_{2\nu+1}-e_{2\nu})/e_{2\nu+1}+e_{2\nu})$ and $\sum_{\nu=1}^{\infty} (e_{2\nu+1}-e_{2\nu})/e_{2\nu}$ diverge simultaneously, L. Myrberg's criterion is the weak condition for the component ∞.

In the case where Γ is not symmetric, it is not clear whether the sufficient conditions of Theorems 5 and 6 are weak for the component ∞ or not.

If they are weak conditions, what is the conditions that is parabolic and unstable for the component ∞?

7. Let us consider the relation between the boundary component of the parabolic type and the concept of a regular and an irregular boundary component in the potential theory.

In order to simplify the observation, we transform the fundamental domain B by $\xi = -\frac{1}{z}$. Then we get the fundamental domain B^* bounded by circles which converge to the origin from the both sides.

We describe any auxiliary closed curve γ_ξ which surrounds the origin and the isolated boundaries of circles and does not intersect them. Let $\omega(\xi)$ be the harmonic function which takes the value 1 on γ_ξ and 0 on the other boundaries contained in γ_ξ.

We say that $\xi=0$ is a regular boundary component, if there exists some neighborhood $U(O)$ of the origin for any $\varepsilon > 0$ so that $\omega(\xi) < \varepsilon$ for each $\xi \in U(O) \cap B^*$, and otherwise $\xi=0$ is an irregular boundary component.

Let $w=f(\xi)$ be a single-valued, regular and univalent function in B^*. The concept of a regular and an irregular boundary component is a local property and conformally invariant. If we denote by γ_w the image of γ_ξ, the harmonic function which takes the value 1 on γ_w and 0 on the other boundaries contained in γ_w is $\overline{\omega}(w)=\omega(f(\xi))$.

Then we have easily the following

Theorem 7, *If the origin $\xi = 0$ is irregular, then it is weak and hence parabolic.*

Proof. Let $\{\gamma_w{}^n\}$, $(n=1, 2, \cdots)$ be a sequence of the closed curves contained in γ_w which converge to the boundary component $\Gamma_w = \lim_{n\to\infty} \gamma_w{}^n$ corresponding to $\xi=0$ and do not intersect the other isolated boundaries.

If we denote by $V_n(w)$ the function which is harmonic in the domain bounded by $\gamma_w{}^n$ and γ_w and takes the value 1 on γ_w and 0 on $\gamma_w{}^n$ and further by $\overline{\omega}_n(w)$ the harmonic function which takes the value 1 on γ_w and 0 on $\gamma_w{}^n$ and the other boundaries, then we have

$$V_n(w) \geqq \overline{\omega}_n(w). \tag{7.1}$$

If $\xi=0$ is unstable, that is, $\xi=0$ corresponds to a continuum Γ_w by some $w=f(\xi)$, both $V_n(w)$ and $\overline{\omega}_n(w)$ converge uniformly to the harmonic functions $V(w)$ and $\overline{\omega}(w)$ which are not constant and we have from (7.1)

$$V(w) \geqq \overline{\omega}(w). \tag{7.2}$$

If we denote by $B_w{}^*$ the image of B^*, since there exists some neighborhood $U(\Gamma_w)$ of Γ_w for any $\varepsilon > 0$ so that $V(w) < \varepsilon$ establishes in $B_w{}^* \cap U(\Gamma_w)$, then we have $\overline{\omega}(w) < \varepsilon$. This fact contradicts with the hypothesis that $\xi=0$ is irregular. q. e. d.

Generally let us consider an infinitely connected domain bounded by a sequence of circles $k_n (n=1, 2, \cdots)$ which do not intersect with each other and converge to $\xi=0$. In this case we have the sufficient conditions for $\xi=0$ to be regular or irregular (L. Myrberg

[7]).

Denote by r_n and a_n the radii of k_n and the distances from $\xi=0$ to the centers of them respectively, where $r_n < a_n$, $a_n < a_{n-1}$ and $\lim_{n\to\infty} a_n=0$. By L. Myrberg's criterion, if the series

$$\sum_{n=1}^{\infty} \frac{\log a_n}{\log r_n} \tag{7.3}$$

converges, then $\xi=0$ is an irregular boundary component.

Then we have the following

Corollary. If the series (7.3) *converges,* $\xi=0$ *is weak and hence parabolic.*

Example $a_n=\frac{1}{n}$, $r_n=\frac{1}{n^{n^k}}(k>1)$.

The converse of Theorem 7 does not necessary establish. Even if $\xi=0$ is parabolic, a regular case may occur.

Counter example. Let $S_n(n=1, 2, \cdots)$ be segments : $x=a_n(<0)$, $|y|\leq h_n$ satisfying $0<a_{n+1}<a_n$, $\lim_{n\to\infty} a_n=0$ and

$$h_n=a_n \tan \alpha \tag{7.4}$$

for some fixed $\alpha\left(0<\alpha<\frac{\pi}{2}\right)$. Then from Theorem 3, $\xi=0$ is a weak boundary component of the domain obtained by deleting $\bigcup_{n=1}^{\infty} S_n \cup \{O\}$ from the ξ-plane. On the other hand we have

$$\lim_{n\to\infty} \frac{\log \frac{a_n}{h_n}}{\log \frac{1}{h_n}}=0 \tag{7.5}$$

and it is the sufficient condition for $\xi=0$ to be regular (L. Myrberg [7]).

From the above and the section 4, an irregular component is the strongest and a parabolic component is the weakest among three concepts.

8. In this section let us seek for the sufficient condition for $\xi=0$ to be non-parabolic. To deal with $\varphi_\nu(\xi)$ in B^* is very difficult in calculation. So we map B^* conformally onto a domain B in the complex z-plane, of which boundaries are slits on the real axis, by Koebe's method which use the series (Koebe [5]). The upper and the lower halves of B are marked with B_0 and B_1 respectively and the notations of the endpoints of slits are the same as B^*.

Let $\xi=k(z)$ be the function which maps B conformally onto B^*, that is, the inverse of Koebe's function. Then instead of considering $\varphi_n(\xi)$ in B^*, we may consider the composed function $\psi_n(z)=\varphi_n(k(z))=U_n(z)+iV_n(z)$ in B. We denote the intervals among an infinite number of slits in order by

$$\begin{cases} b_1(\infty, e_2),\ b_2(e_3, e_4), \cdots, b_\nu(e_{2\nu-1}, e_{2\nu}), \cdots\cdots \\ b_{-1}(-\infty, e_{-2}),\ b_{-2}(e_{-3}, e_{-4}), \cdots, b_{-\nu}(e_{-(2\nu-1)}, e_{-2\nu}), \cdots\cdots. \end{cases} \tag{8.1}$$

We define a single-valued, bounded and harmonic function $\overline{V}_\nu(z)$ or $\overline{V}_{-\nu}(z)$ in a domain B_ν or $B_{-\nu}$ bounded by only two slits (b_ν, b_1) or $(b_{-\nu}, b_{-1})$ which satisfy the following condition :

$$\overline{V}_{\pm}(z)=\begin{cases}0, & \text{on } b_{\pm 1}\\ \pi, & \text{on } b_{\pm\nu}.\end{cases} \tag{8.2}$$

Evidently we have the following inequality

$$V_n(z) < \sum_{\nu=n+1}^{\infty} (\overline{V}_\nu(z)+\overline{V}_{-\nu}(z)). \tag{8.3}$$

In order to get the value $V_n(z)$ in the neighborhnnd of $z=0$, we may evaluate $\overline{V}_\nu(z)$ and $\overline{V}_{-\nu}(z)$ there. For this purpose we make use of the elliptic integral. For converience we transform the variable in the following manner. At the first we carry out $\xi=z^2$ in B_ν or $B_{-\nu}$ and the second we do the inverse $\eta=\frac{1}{\xi}$ in the transformed ξ-plane. Since

$$\overline{V}_\nu^*(\eta)=\overline{V}_\nu(z),\quad \overline{V}_{-\nu}^*(\eta)=\overline{V}_{-\nu}(z) \tag{8.4}$$

in η-plane, we can calculate $\overline{V}_\nu^*(\eta)$ and $\overline{V}_{-\nu}^*(\eta)$, if we choose the constant c_ν so that the period of the elliptic integral is π in the following :

$$\overline{V}_\nu^*(-\infty)=c_\nu\int_0^{-\infty}\frac{dx}{\sqrt{x(x-a_2)(x-a_{2\nu-1})(x-a_{2\nu})}}, \tag{8.5}$$

$$\overline{V}_\nu^*(+\infty)=\pi+c_\nu\int_{a_{2\nu}}^{\infty}\frac{dx}{\sqrt{x(x-a_2)(x-a_{2\nu-1})(x-a_{2\nu})}}, \tag{8.6}$$

where $\eta=x+iy$ and $a_{2\nu}=\frac{1}{e^2_{2\nu}}$ and $a_{2\nu-1}=\frac{1}{e^2_{2\nu-1}}$. Since

$$\overline{V}_\nu^*(-\infty)=\overline{V}_\nu^*(+\infty), \tag{8.7}$$

then we have

$$\left|c_\nu\int_0^{-\infty}\frac{dx}{\sqrt{}}\right|+\left|c_\nu\int_{a_{2\nu}}^{\infty}\frac{dx}{\sqrt{}}\right|=\pi. \tag{8.8}$$

We use a similar estimation as in L. Myrberg ([6]).

Suppose that

$$a_{2\nu}=a_{2\nu-1}+d_\nu,\quad (d_\nu<1). \tag{8.9}$$

Then we have

$$\frac{\left|c_\nu\int_0^{-\infty}\frac{dx}{\sqrt{}}\right|}{\left|c_\nu\int_{a_{2\nu}}^{\infty}\frac{dx}{\sqrt{}}\right|}\leqq k\,\frac{\log a_{2\nu-1}}{|\log d_\nu|}, \tag{8.10}$$

where k is a constant independent of ν. From (8.8) and (8.10), we have

$$(8.11)\qquad \left| c_\nu \int_0^{-\infty} \frac{dx}{\sqrt{\qquad}} \right| \leqq \pi k \frac{\log a_{2\nu-1}}{|\log d_\nu|}.$$

Similarly to the calculation of $\overline{V}_{-\nu}(z)$ we have

$$(8.12)\qquad \left| c_\nu' \int_0^{-\infty} \frac{dx}{\sqrt{\qquad}} \right| \leqq \pi k' \frac{\log a_{-(2\nu-1)}}{|\log d_{-\nu}|},$$

where k' and $d_{-\nu}$ are constant as in (8.9) and (8.10). From (8.11) and (8.12) the following inequality

$$(8.13)\qquad V_n(0) < \pi\Big[k^* \sum_{\nu=n+1}^{\infty} \Big(\frac{\log a_{2\nu-1}}{|\log d_\nu|} + \frac{\log a_{-(2\nu-1)}}{|\log d_{-\nu}|} \Big)\Big]$$

establishes, where $k^* = \max(k,k')$. Under the condition

$$(8.14)\qquad \sum_{\nu=n+1}^{\infty} \Big(\frac{\log a_{2\nu-1}}{|\log d_\nu|} + \frac{\log a_{-(2\nu-1)}}{|\log d_{-\nu}|} \Big) < \frac{1}{k^*}$$

we have

$$(8.15)\qquad V_n(0) < \pi$$

and hence $z=0$ is non-parabolic.

Suppose that the series

$$(8.16)\qquad \sum_{\nu=2}^{\infty} \Big(\frac{\log a_{2\nu-1}}{|\log d_\nu|} + \frac{\log a_{-(2\nu-1)}}{|\log d_{-\nu}|} \Big)$$

converges. Then taking enough a great number $n_0(k^*)$, for $n \geqq n_0(k^*)$, (8.14) and hence (8.15) establish.

If we return to the z-plane, (8.16) is transformed into

$$(8.17)\qquad \sum_{\nu=2}^{\infty} \Big(\frac{\log 1/e^2_{2\nu-1}}{|\log(1/e^2_{2\nu} - 1/e^2_{2\nu-1})|} + \frac{\log 1/e^2_{-(2\nu-1)}}{|\log(1/e^2_{-2\nu} - 1/e^2_{-(2\nu-1)})|} \Big).$$

To simplify the form of (8.17), if we denote $1/e_{2\nu-1}$ and $1/e_{-(2\nu-1)}$ be $p_{2\nu-1}$ and $p_{-(2\nu-1)}$ respectively, we have

$$(8.18)\qquad 2 \sum_{\nu=2}^{\infty} \Big(\frac{\log p_{2\nu-1}}{|\log(p^2_{2\nu} - p^2_{2\nu-1})|} + \frac{\log p_{-(2\nu-1)}}{|\log(p^2_{-2\nu} - p^2_{-(2\nu-1)})|} \Big).$$

Under the hypothesis $p^2_{2\nu} - p^2_{2\nu-1} < 1$ and $p^2_{-2\nu} - p^2_{-(2\nu-1)} < 1$, (8.18) and

$$(8.19)\qquad \sum_{\nu=2}^{\infty} \Big(\frac{\log p_{2\nu-1}}{|\log(p_{2\nu} - p_{2\nu-1})|} + \frac{\log p_{-(2\nu-1)}}{|\log(p_{-2\nu} - p_{-(2\nu-1)})|} \Big)$$

are simultaneously convergent. Then we have the following

Theorem 8. *Under the hypothesis that* $(p_{2\nu} - p_{2\nu-1})$, $(p_{-2\nu} - p_{-(2\nu-1)}) < 1$, *if* (8.19) *is convergent, then the component* $z=0$ *is non-parabolic and hence unstable.*

In the special case of Theorem 8 when Γ is a symmetric group, we have the following

Corollary. (L. Myrberg [6]). *If*

$$(8.20)\qquad \sum_{\nu=2}^{\infty} \frac{\log p_{2\nu-1}}{|\log(p_{2\nu} - p_{2\nu-1})|}$$

converges under $p_{2\nu}-p_{2\nu-1}<1$, *the component* $z=0$ *is non-parabolic and hence unstable.*

example. $p_{2\nu-1}=\nu,\ p_{2\nu}-p_{2\nu-1}=\frac{1}{\nu^{\nu^k}}\ (k>1)$.

Remark. Oikawa proved the following ([13], Theorem 8) :

Let $S_\nu(\nu=1, 2, \cdots)$ be a sequence of closed intervals $[e_{2\nu+1}, e_{2\nu}]$ on the positive real axis of the complex z-plane, where $0<e_{2\nu+1}<e_{2\nu}<e_{2\nu-1}<1$ $(\nu=2, 3, \cdots)$ and $\lim_{\nu\to\infty} e_{2\nu}=0$. Then, under the conditions

$$\lim_{\nu\to\infty}\frac{e_{2\nu-1}}{e_{2\nu}}=1 \tag{8.21}$$

and

$$\frac{e_{2\nu-1}}{e_{2\nu+1}}\geqq 1+\delta>1, \tag{8.22}$$

the component $z=0$ is weak if and only if

$$\sum_{\nu=1}^{\infty}\frac{1}{\log\dfrac{e_{2\nu}}{e_{2\nu-1}-e_{2\nu}}}=\infty. \tag{8.23}$$

But the above Corollary is modified to the following :

Under the condition

$$\frac{e_{2\nu-1}-e_{2\nu}}{e_{2\nu}\ e_{2\nu-1}}<1, \tag{8.24}$$

if

$$\sum_{\nu=1}^{\infty}\frac{|\log e_{2\nu-1}|}{\log\dfrac{e_{2\nu}}{e_{2\nu-1}-e_{2\nu}}}<\infty, \tag{8.25}$$

then the component $z=0$ is non-parabolic.

It is easily seen that (8.24) is stronger than (8.21). Then from the above we find that under the conditions (8.22) and (8.24), if the component $z=0$ is parabolic and unstable, (8.25) is divergent, but (8.23) is convergent.

References

[1] Akaza, T., : A property of some Poincaré theta-series, Nagoya Math. Jour., 16 (1960), 189-194.

[2] Akaza, T., : On the weakness of some boundary component, ibid. 17 (1960), 219-223.

[3] Akaza, T. and Oikawa, K., : Examples of weak boundary component, ibid., 18 (1961), 165-170.

[4] Burnside, W., On a class of automorphic functions, Proc. London Math. Soc., 23 (1891), 49-88.

[5] Koebe, P., : Über die konforme Abbildung endlich-und unendlichvielfach zusammenhängender symmetrischer Bereiche, Acta Math., 43 (1922), 263-287.

[6] Myrberg, L., : Normalintegral auf zweiblättrigen Riemannschen Flächen mit reellen Verzweigungspunkten, Ann. Acad. Sci. Fennicae., A-I 71 (1950), 1-50.

[7] Myrberg, L., : Über reguläre und irreguläre Randpunkte des harmonischen Masses, ibid., 91 (1951), 1-11.

[8] Myrberg, P. J., : Über die Bestimmung des Typus einer Riemannschen Fläche, Ann. Acad. Sci. Fennicae., A XLV, **3** (1935), 1-30.

[9] Myrberg, P. J., : Über transzendente hyperelliptische Integrale erster Gattung, Ann. Acad. Sci. Fennicae., A-I. **14** (1943), 1-32.

[10] Myrberg, P. J., : Über analytische Funktionen auf transzendenten zweiblättrigen Riemannschen Flächen mit reellen Verzweigungspunkten, Acta Math., **76** (1944), 184-224.

[11] Myrberg, P. J., : Über Integrale auf transzendenten symmetrischen Riemannschen Flächen, Ann. Acad. Sci. Fennicae, A-I **31** (1945), 1s-20.

[12] Myrberg, P. J., : Über analytische Funktionen auf transzendenten Riemannschen Flächen, X. Congr. Math. Scand., Copenhague, (1946), 77-96.

[13] Oikawa, K., : On the stability of boundary components, Pacific Jour. Math., **10** (1960), 263-294.

[14] Sario, L., : Strong and weak boundary components, Jour. Anal. Math., **5** (1956/57), 389-398.

[15] Savage, N., : Weak boundary components of an open Riemann surface, Duke Math. Jour., **24** (1957), 79-95.