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The Law of the Iterated Logarithms

By

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1. Introduction. The purpose of this note is to prove the following

Theorem. Let $f(t)$ be a function satisfying the conditions : $f(t+1)=f(t)$, $\int_0^1 f(t)dt=0$, $\int_0^1 f^2(t)dt=1$ and for some $\alpha>0$,

(1.1) $$\left[\int_0^1 (f(t)-S_n(t))^2\,dt\right]^{1/2}=0(n^{-\alpha})$$

as $n\to+\infty$, where $S_n(t)$ denotes the n-th partial sum of the Fourier series of $f(t)$. Then if a sequence of positive integers $\{n_k\}$ satisfies

(1.2) $$\frac{n_{k+1}}{n_k}\geqq 4\,[\log^c(k+2)],$$

where c is a positive number such that

(1.3) $$2\,\alpha\,c>1,$$

we have, for almost all t,

(1.4) $$\varlimsup_{N\to\infty}\frac{1}{\sqrt{2N\log_2 N}}\sum_{k=1}^{N} f(n_k t)=1.$$

In [1] M. Weiss proved that the law of the iterated logarithms holds for a lacunary trigonometric series. However (1.4) does not hold even for the case where $f(t)$ is a trigonometric polynomial and $\{n_k\}$ satisfies the Hadamard's gap condition.

2. Some Lemma. For simplicity we put

(2.1) $$f(t)\sim\sum_{l=1}^{\infty}c_l\cos 2\pi lt,$$

then we have

(2.1′) $$g_k(t)=f(t)-S_{\mu_k^3}(t)\sim\sum_{l>\mu_k^3}c_l\cos 2\pi lt$$

where

(2.1″) $$\mu_k=[\log^c(k+2)].$$

Lemma 1. We have, for almost all t,

$$\lim_{N\to\infty}\frac{1}{\sqrt{N}}\sum_{k=1}^{N}(f(n_k t)-S_{\mu_k}(n_k t))=0.$$

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Proof. In the following A will denote an absolute constant. From (1.2), we have

$$\frac{\mu_{k+1}\, n_{k+1}+1}{\mu_k^2\, n_k} \geqq 4\, \frac{\mu_{k+1}}{\mu_k} > 4. \tag{2.2}$$

Hence $\{S_{\mu_k^2}(n_k t)-S_{\mu_k}(n_k t)\}$ is a system of orthogonal functions and by (1.1), (1.2) and (1.3), we have

$$\int_0^1 \Big[\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \Big(S_{\mu_k^2}(n_k\, t)-S_{\mu_k}(n_k\, t)\Big)\Big]^2 dt$$

$$\leqq \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \int_0^1 (f(t)-S_{\mu_k}(t))^2\, dt \leqq \sum_{k=1}^{\infty} \frac{A}{k(\log\, k)^{2\alpha c}} < \infty.$$

This implies that the series

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}\Big(S_{\mu_k^2}(n_k\, t)-S_{\mu_k}(n_k\, t)\Big) \tag{2.3}$$

is the Fourier series of a square integrable function. Therefore by (2.2) and a theorem of A. N. Kolmogorov, the series (2.3) converges almost everywhere in t. Hence we have, for almost all t,

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \Big(S_{\mu_k^2}(n_k\, t)-S_{\mu_k}(n_k\, t)\Big) =0. \tag{2.4}$$

In the same way we have, for almost all t,

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \Big(S_{\mu_k^2}(n_k\, t)- S_{\mu_k^3}(n_k\, t)\Big) =0. \tag{2.5}$$

For any integers k and k' such that $2^N \leqq k < k' \leqq 2^{N+1}$, we have

$$I(k,\, k') = \Big|\int_0^1 g_k(n_k\, t) g_{k'}(n_{k'}\, t)dt\Big| = \Big|\frac{1}{2} \sum_{l>\mu_{k'}^3} c_l\, d_l(k,\, k')\Big|,$$

where

$$d_l(k,\, k')=\begin{cases} c_l \dfrac{n_{k'}}{n_k}, & \text{if } n_k \mid l n_{k'}, \\ 0\quad, & \text{if otherwise.} \end{cases}$$

Hence we have, by (1.1), (1.2) and (1.3),

$$I(k,\, k')\leqq \frac{1}{2}\Big(\sum_{l>\mu_{k'}^3} c_l^2\Big)^{1/2}\Big(\sum_{l>\mu_{k'}^3} d_l^2\,(k,\, k')\Big)^{1/2}\leqq \frac{A}{N^{6\alpha c}\, 4^{\alpha(k'-k)}}$$

From the above relation, it follows that for $2^N \leqq m < m' \leqq 2^{N+1}$

$$\int_0^1 \Big(\sum_{k=m}^{m'} g_k(n_k\, t)\Big)^2 dt = \sum_{k=m}^{m'} \int_0^1 g_k^2(t)\, dt+2 \sum_{k=m}^{m'-1} \sum_{k'>k}^{m'} I(k,\, k')\leqq \frac{A(m'-m)}{N^{6\alpha c}}$$

By the well known device of D. Menchoff, we have

$$\int_0^1 \max_{2^N\leqq m\leqq 2^{N+1}} \Big|\frac{1}{\sqrt{2^N}} \sum_{k=2^N}^{m} g_k(n_k\, t)\Big|^2 dt \leqq \frac{AN^2}{N^{6\alpha c}}$$

Since $2\alpha c > 1$, we have

$$\sum_N \int_0^1 \max_{2^N \leq m \leq 2^{N+1}} \left(\frac{1}{\sqrt{2^N}} \sum_{k=2^N}^{m} g_k(n_k\, t) \right)^2 dt < \infty,$$

and this implies, for almost all t,

$$\lim_{N \to \infty} \max_{2^N \leq m \leq 2^{N+1}} \frac{1}{\sqrt{2^N}} \sum_{k=2^N}^{m} g_k(n_k\, t) = 0.$$

Hence we have, for almost all t,

$$\lim_{N \to \infty} \frac{1}{\sqrt{N}} \sum_{k=1}^{N} g_k(n_k\, t) = 0. \tag{2.6}$$

(2.4), (2.5) and (2.6) prove the lemma.

3. Fundamental Inequalities. we put, for $k=1, 2, \cdots$,

$$b_k^2 = \int_0^1 S_{\mu_k}^2(n_k\, t)dt \qquad \text{and} \qquad B_k^2 = \sum_{j=1}^{k} b_j^2.$$

Then we have, for all $k \leq N$,

$$| S_{\mu_k}(n_k\, t) | \leq \sum_{l=1}^{\mu_k} | c_l | \leq (2 b_k^2 \mu_k)^{1/2} \leq (2\mu_N)^{1/2}. \tag{3.1}$$

Lemma 2. Let λ be a positive number and I be any inerval in [0,1] such that

$$(2\mu_N)^{1/2} \lambda \leq \frac{1}{3}, \tag{3.2}$$

$$| I | > \frac{2}{n_1}, \tag{3.2'}$$

then, for any interval I contained in [0,1], we have

$$\frac{| I |}{2} \exp\left(\frac{\lambda^2 B_N^2}{2} (1-\eta) \right) \leq \int_I \exp\left(\lambda \sum_{k=1}^{N} S_{\mu_k}(n_k\, t) \right) dt \tag{3.3}$$

$$\leq \frac{3}{2} | I | \exp\left(\frac{\lambda^2 B_N^2}{2} (1+\eta) \right),$$

where η is a constant satisfying

$$0 \leq \eta \leq 12 \lambda \eta_N^{3/2}. \tag{3.4}$$

Proof. The proof is based on the inequality :

$$\left| \log\left(1+z+\frac{1}{2}z^2 \right) - z \right| \leq 2 | z |^3, \qquad \text{for } | z | < 1/3.$$

We have, by (3.1), (3.2) and the above ineqality,

$$\exp(\lambda S_{\mu_k}(n_k\, t)) = (1+\lambda S_{\mu_k}(n_k\, t) + \frac{\lambda^2}{2} S_{\mu_k}^2(n_k\, t))\, e^{Q_k(t)}, \tag{3.5}$$

and, by (3.1),

$$\sum_{k=1}^{N} | Q_k(t) | \leq 2 \sum_{k=1}^{N} | \lambda S_{\mu_k}(n_k\, t) |^3 \leq \lambda^2 B_N^2 (2^{5/2} \lambda \mu_N^{3/2}). \tag{3.5'}$$

Now if we write

$$(3.6)\qquad 1+\lambda\, S_{\mu_k}(n_k\, t)+\frac{1}{2}\lambda^2\, S^2_{\mu_k}(n_k\, t)=\left(1+\frac{\lambda^2\, b^2_{\mu_k}}{2}\right)+T_k(t),$$

then $T_k(t)$ is the sum of non-constant terms and we have

$$(3.6')\qquad T_k(t)=\sum_{l=1}^{2\mu_k} d_l^{(k)}\cos\ 2\pi l n_k t,$$

and, by (3.2),

$$(3.6'')\qquad \sum_{l=1}^{2\mu_k} |\, d_l^{(k)}\,| \leqq \lambda \sum_{l=1}^{\mu_k} |\, c_l\,| +\frac{\lambda^2}{2}\left(\sum_{l=1}^{\mu_k} |\, c_l\,|\right)^2$$

$$\leqq \sqrt{2\,\mu_k}\,\lambda+\frac{1}{2}(\sqrt{2\,\mu_k}\,\lambda)^2 \leqq \frac{1}{3}+\frac{1}{18}=\frac{7}{18}.$$

Putting

$$(3.7)\qquad P_N(t)=\prod_{k=1}^{N}\left(1+\frac{\lambda^2\, b_k^2}{2}+T_k(t)\right),$$

it is seen that

$$(3.7')\qquad P_N(t)=\prod_{k=1}^{N}\left(1+\frac{\lambda^2\, b_N^2}{2}\right)$$

$$+\sum_{k=2}^{N}\left\{\prod_{N\geqq j>k}\left(1+\frac{\lambda^2\, b_j^2}{2}\right)\right\} T_k(t)\left\{\prod_{j<k}\left(1+\frac{\lambda^2\, b_j^2}{2}+T_j(t)\right)\right\}+\prod_{j=2}^{N}\left(1+\frac{\lambda^2\, b_j^2}{2}\right)T_1(t).$$

Since (1.2) implies $n_k-2\sum_{j=1}^{k-1}\mu_j n_j > n_k/3$, we have

$$(3.8)\qquad \left|\int_I \prod_{j=1}^{k} \cos\, 2\pi l_j n_j t\ dt\right| \leqq \frac{6}{n_k} \leqq \frac{1}{n_1 4^{k-1}},$$

where $0\leqq l_j \leqq 2\mu_j$ for $j<k$ and $1\leqq l_k$.

Therefore if we put

$$\int_I P_N(t)dt=|\,I\,|\prod_{k=1}^{N}\left(1+\frac{\lambda^2\, b_k^2}{2}\right)+R_N,$$

we have, by (3.7′), (3.8), (3.6′′) and (3.2′)

$$|\,R_N\,|\leqq \prod_{k=1}^{N}\left(1+\frac{\lambda^2\, b_k^2}{2}\right)\left\{\sum_{k=2}^{N}\left|\int_I T_k(t)\prod_{j<k}\left(1+\frac{\lambda^2\, b_j^2}{2}+T_j(t)\right)dt\right|\right.$$

$$\left.+\left|\int_I T_1(t)\,dt\right|\right\}$$

$$\leqq \frac{1}{n_1}\prod_{k=1}^{N}\left(1+\frac{\lambda^2\, b_k^2}{2}\right)\left\{\sum_{k=2}^{N}\frac{\sum_{l=1}^{2\mu_k}|\, d_l^{(k)}|}{4^{k-1}}\prod_{j=1}^{k-1}\left(1+\frac{\lambda^2\, b_j^2}{2}+\sum_{l=1}^{2\mu_j}|\, d_l^{(j)}\,|\right)\right.$$

$$\left.+\sum_{l=1}^{2\mu_1}|\, d_l^{(1)}\,|\right\}$$

$$\leqq \frac{2}{n_1} \prod_{k=1}^{N} \left(1+\frac{\lambda^2 b_k^2}{2}\right) \frac{7}{18} \left\{ \sum_{k=2}^{N} \left(\frac{1}{4^{k-1}}\right) \left(\frac{51}{36}\right)^{k-1} + \frac{1}{3} \right\} \leqq \frac{|I|}{2} \prod_{k=1}^{N} \left(1+\frac{\lambda^2 b_k^2}{2}\right).$$

Hence we have

$$\int_I P_N(t)dt \leqq \frac{3|I|}{2} \prod_{k=1}^{N} \left(1+\frac{\lambda^2 b_k^2}{2} \leqq \frac{3|I|}{2} \exp\left(\frac{\lambda^2 b_N^2}{2}\right)\right. \tag{3.9}$$

and, by (3.2)

$$\int_I P_N(t)dt \geqq \frac{|I|}{2} \prod_{k=1}^{N} \left(1+\frac{\lambda^2 b_k^2}{2}\right) \geqq \frac{|I|}{2} \exp\left(\sum_{k=1}^{N} \frac{\lambda^2 b_k^2}{2} - \sum_{k=1}^{N} \frac{\lambda^4 b_k^4}{4}\right) \tag{3.9'}$$

From (3.5), (3.5′), (3.9) and (3.10) we obtain the second half of (3.3) and from (3.5), (3.5′), (3.9) and (3.10′) the first half of (3.3).

This lemma corresponds to the Lemma 1 of [1]. Using this lemma and the fact that

$$B_N^2 - N = O\left(\frac{N}{\mu_N{}^{2\alpha}}\right) \qquad \text{as } N \to +\infty,$$

we can prove that, in the same way as that of [1],

$$\overline{\lim_{N\to\infty}} \frac{1}{\sqrt{2N \log_2 N}} \sum_{k=1}^{N} S_{\mu_k}(n_k t) = 1 \tag{3.11}$$

holds for almost all t. By (3.11) and Lemma 1 we can prove the theorem.

Reference

[1] M. Weiss, On the law of the iterated logarithms for lacunary trigonometric series, Trans. Amer. Math. Soc., vol. 91 (1959) pp. 444-469.