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A New Characterization of the Sphere and the Isoperimetric Problem in E_3

By

Shigeru OHSHIO*

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Introduction

The extremal geometric problems are usually expressed in the form of inequalities stated in terms of geometric concepts. For example, the isoperimetric problem in the three-dimensional Euclidean space E_3 is expressed by the isoperimetric inequality

$$S^3-36\pi V^2 \geqq 0.$$

Almost all of them can be solved by the methods of the calculus of variations. However these methods are cumbersome and extremely laborious to handle, especially in E_3. Therefore other methods are given in this branch of geometry, for example, the various symmetrizations, and linear combination of convex bodies. The latter methods are more elegant and more precise than those of the calculus of variations. But in general these methods can be applied only to small selection of special problems.

Therefore it is more desirable to establish differential and integral formulas for the quantities associated with convex bodies. For this purpose we shall establish a set of differential formulas and a corresponding set of integral formulas for such quantities.

In a preceding paper [9][1] we treated the problem in a case when the total mean curvature $M^*(t)$ is continuous over the whole interval $0 \leqq t \leqq r$ and the characteristic function $\varkappa(t)$ is a step-function. When the total mean curvature $M^*(t)$ is discontinuous and the characteristic function $\varkappa(t)$ is a monotone decreasing step-function, the problems have been definitely shown in the paper [10].

The remaining problem is to determine completely the characteristic function $\varkappa(t)$ in the general case. The main purpose of this paper is to show the general types of the characteristic function $\varkappa(t)$ and to generalize the results of the preceding papers [9] and [10]. Among our results in this paper, it should be especially noted that the value of the characteristic function $\varkappa(t)$ takes any real number.

We begin with a set of interior parallel convex polyhedra. First, we shall give the definition of the sequence of interior parallel polyhedra of a convex polyhedron. In order

* Department of Mathematics, Kanazawa University.

1) Numbers in brackets refer to the references at the end of the paper.

to make the change of faces of a polyhedron of the sequence clear, we shall prove the first decomposition theorem. In regard to the change of an edge of the sequence, we shall give a new definition of a negative edge. The sequence which contains some of the negative edges is to be called edge-singular. Then we shall prove the second decomposition theorem. Further it will be proved that the form-figure of the edge singular sequence of the interior parallel polyhedra may be concave and its characteristic function may take any real number. At the end of §1, we shall establish a set of differential formulas associated with a sequence of interior parallel polyhedra.

In §2, in order to extend the results obtained in §1 for the sequence of interior parallel polyhedra to the case of the sequence of interior parallel bodies, first we shall establish a set of approximation theorems. Especially the third approximation theorem definitely shows that a closed bounded convex set with interior points can be approximated by such a sequence of convex polyhedra which are defined as an intersection of the closed half-spaces bounded by its extreme supporting planes. On the base of such an approximation, we shall extend the results obtained in §1 to all the cases of the interior parallel sequences of closed bounded convex bodies and establish a set of differential formulas and a corresponding set of integral formulas associated with the quantities of the sequence.

In §3, employing the preceding results, we shall give a new characterization of the sphere and prove a set of isoperimetric inequalities. Finally we shall give the integral representations of the isoperimetric deficiencies and obtain a set of isoperimetric inequalities which contain the integral concerning the quantities.

These results can be directly extended to the case of the relative differential geometry in the n-dimensional space.

§ 1. Interior parallel convex polyhedra

1.1. The definition of an interior parallel sequence

Let P be a closed bounded convex polyhedron with interior points in the three-dimensional Euclidean space E_3. The faces of P we denote by $A^1, \cdots, A^f$. The class of spheres contained in P have radii which form a bounded set of numbers. By the Blaschke selection theorem there is a sphere of this class whose radius is equal to the upper bound of this set. Such a sphere is called an *insphere* of P and its radius the *inradius* of P. P may have more than one insphere but its inradius is unique. The set of all the centres of the inspheres of P forms a point, a line-segment of finite length or a convex polygon of finite area. Such a set is called a *point-kernel*, a *line-kernel* or a *plane-kernel* of P respectively. The plane containing the face A^i is denoted by $\overline{A}^i$. We define the closed negative half-space $\widetilde{A}^i$ as the half-space which is bounded by $\overline{A}^i$ and contains P. Let us denote the plane parallel to $\overline{A}^i$ and lying at a distance $\tau(\geqq 0)$ from it within $\widetilde{A}^i$ by $\overline{A}^i[\tau]$, and the closed negative half-space $\widetilde{A}^i[\tau]$ as the half-space bounded by $\overline{A}^i[\tau]$ within $\widetilde{A}$. We define the interior parallel polyhedron of P at the distance τ to mean the intersection of all the closed negative half-spaces $\widetilde{A}^i[\tau]$: $i=1,\cdots,f$ and denote it by $P[\tau]$. Or we have

$$P[\tau]=\bigcap_{i=1}^{f}\widetilde{A}^i[\tau].$$

If τ is sufficiently small, the interior parallel polyhedron $P[\tau]$ is non-empty. If $P[\tau]$ is non-empty, each of the faces of $P[\tau]$ corresponds to a face of P whose distance from

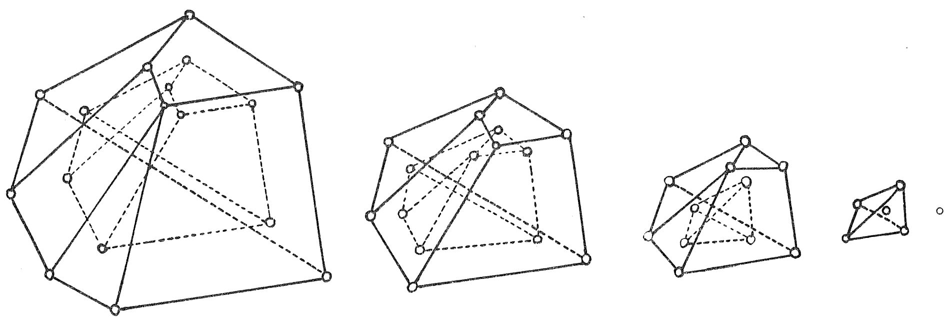

Fig. 1

the face of $P[\tau]$ is τ and which is parallel to it. But the inverse correspondence is not in general preserved, since as τ increases some of faces of $P[\tau]$ vanish. On the other hand, as τ increases the numbers of the edges and the vertices of $P[\tau]$ may decrease or increase (see Fig. 1).

Hence the correspondence between the edges of $P[\tau]$ and P is not in general one-to-one. We denote the inradius of P by r. If τ tends to r, the interior parallel polyhedron $P[\tau]$ converges to the kernel of P. For any $\tau>r$, the set $P[\tau]$ is empty. Thus the interior parallel polyhedron $P[\tau]$ of a closed bounded convex polyhedron P with the insphere of radius r is defined in the interval $0\leqq\tau<r$. For convenience of the following treatment we adopt a new parameter t whose value is equal to $r-\tau$. Then we can denote the interior polyhedron $P[r-\tau]$ of P at the distance $r-\tau$ by $P(t)$. Hence the original polyhedron P is expressed by $P(r)$ and the kernel of P is denoted by $P(0)$.

Including the kernel $P(0)$, we denote the *sequence of interior parallel-surface* $P(t)$ *of* P *with the insphere of radius* $r(>0)$ by $\{P(t)\ :\ 0\leqq t\leqq r\}$. For the value t that $0<t\leqq r$, we denote the faces of $P(t)$ by $A^i(t)$; $i=1,\cdots,f(t)$, its edges by $B^j(t)$; $j=1,\cdots,e(t)$ and its vertices by $C^k(t)$; $k=1,\cdots,v(t)$ respectively. Then the numbers $f(t)$, $e(t)$ and $v(t)$ change their values as the parameter t decreases. But since the Euler's formula for polyhedra holds, we have

$$f(t)-e(t)+v(t)=2, \qquad 0<t\leqq r.$$

First, we investigate the change of the faces of $P(t)$ as the parameter t decreases from r to zero.

1.2. The decomposition theorem I

i) The inner body to a face A^i

The value of the parameter t of the sequence $\{P(t) : 0 \leqq t \leqq r\}$ which corresponds to the vanishing of a face or an edge of $P(t)$ will be called the *critical values* of the parameter. We denote such a critical value as corresponds to the vanishing of the face $A^i(t)$ by $\rho(A^i)$. Let v be the number of the faces which have the edges or the vertices in common with the face $A^i(t)$ as t decreases from r to $\rho(A^i)$. Renumbering such v faces, we denote them by

$$A^{i_1}, \cdots, A^{i_v}.$$

Suppose that the plane $\bar{\Lambda}^{i_k}$ bisects the dihedral angle which is formed by the planes $\bar{A}^{i}$ and $\bar{A}^{i_k}$ and meet the polyhedron P, $(k=1, \cdots, v)$. Let $\tilde{\Lambda}^{i_k}$ denote the closed half-space bounded by the plane $\bar{\Lambda}^{i_k}$ which contains the face A^i, $(k=1, \cdots, v)$ (see Fig. 2).

Then the intersection of the closed half-space $\tilde{A}^i$ and the intersection $\bigcap_{k=1}^{v} \tilde{\Lambda}^{i_k}$ is called the *inner body to the face* A^i and is denoted by $I(A^i)$. Thus we have

$$I(A^i) = \left(\bigcap_{k=1}^{v} \tilde{\Lambda}^{i_k} \right) \cap \tilde{A}^i.$$

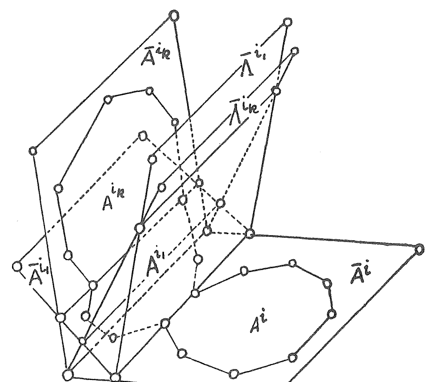

Fig. 2

ii) The decompostition theorem

Theorem 1. *A closed convex polyhedron P with interior points is decomposed into the inner bodies to faces $I(A^i) : i=1, \cdots, f$, of the polyhedron. Thus we have*

$$P = I(A^1) \cup \cdots \cup I(A^f). \tag{1}$$

Proof. If X is any point of P, X is an interior point or a frontier point of P. If X is an interior point of P, the sphere whose centre is X and whose radius is sufficiently small is contained in P and, on the other hand, the sphere whose centre is X and whose radius is sufficiently large contains P. Hence the set of the concentric spheres whose common centre is X is divided into two subset: one consisting of the spheres containing only the interior points of P and the other consisting of the spheres each of which contains at least one exterior point of P. Let the superior limit of the radii of the spheres which belong to the former be denoted by s. If $\mathfrak{S}(X, s)$ denotes the sphere whose centre is X and whose radius is s, the sphere $\mathfrak{S}(X, s)$ touches at least one face of P. First, suppose the case where $\mathfrak{S}(X,s)$ touches only a face A^i. Then the distance of the face A^i from the point X is s, but the distances of the other faces of P from X are greater than s. Thus the point X is the interior point of the inner body $I(A^i)$ to the face A^i.

Suppose the case in which the sphere $\mathfrak{S}(X, s)$ touches faces $A^i, A^j, A^k, \cdots$. Then the distances of the faces $A^i, A^j, A^k, \cdots$ from X are equal to s. Thus since the point X is contained by the planes which bisect the dihedral angles formed by the plane $\overline{A}^i$ and each of the planes $\overline{A}^j$, $\overline{A}^k, \cdots$, the point X is common to the inner bodies $I(A^i)$, $I(A^j)$, $I(A^k), \cdots$.

Next, when the point X is a frontier point of P, X belongs to a face only or two or more faces.

Thus the point X which belongs to P is contained by one of f inner bodies $I(A^1), \cdots$, $I(A^f)$ or the frontier point in common to two or more of them. Since the f inner bodies $I(A^1), \cdots, I(A^f)$ have no interior points in common to each other and cover the polyhedron P leaving no space, the theorem has been proved.

iii) The relative inradius of a face A^i

Since the inner body $I(A^i)$ to the face A^i is bounded and closed, the distances of the face A^i from points which belong to $I(A^i)$ form a bounded set of numbers. By the Blaschke selection theorem there is an upper bound of these values. If we denote such a value by $d(A^i)$, there is a set of points which belong to $I(A^i)$ and whose distances from the face A^i are equal to $d(A^i)$ and the set of such points forms a part of the boundary of $I(A^i)$. It is easy to see that the sphere whose centre is a point of such a set and whose radius is $d(A^i)$ touches the face A^i and is contained in the polyhedron P.

Definition. A set of the frontier points of the inner body $I(A^i)$ to a face A^i whose distances from A^i are equal to $d(A^i)$ is called the *face-centre* of the face A^i. The sphere whose centre is a point of the face-centre of A^i and whose radius is $d(A^i)$ is called a ***relative insphere*** to the face A^i and its radius $d(A^i)$ is called a ***relative inradius*** of the face A^i.

Suppose that $r-d(A^i) < t \leqq r$. Then the distance of the face $A^i(t)$ from the face-centre of A^i is equal to $t-(r-d(A^i))(>0)$. Hence $P(t)$ contains the face $A^i(t)$. If $t < r-d(A^i)$, then $P(t)$ does not contain such a face as corresponds to the face A^i of P. In other words, the face $A^i(t)$ of the interior parallel polyhedron $P(t)$ vanishes when $t \to (r-d(A^i))+0$ and the limiting figure of the vanishing face $A^i(t)$ is the face-centre of the face A^i. Thus we have

Theorem 2. *If $d(A^i)$ is the relative inradius of the face A^i of a convex polyhedron P, the critical value $\rho(A^i)$ which corresponds to the vanishing of the face $A^i(t)$ of the polyhedron $P(t)$ $(0 \leqq t \leqq r)$ is equal to $r-d(A^i)$.*

1.3. Sets and number associated with an edge

i) The definition

Let $B^k(t)$ be an edge of $P(t)$ which is the intersection of the faces $A^i(t)$ and $A^j(t)$, and let $C^m(t)$ and $C^n(t)$ be the extremities of the edge $B^k(t)$. We denote the faces which have the vertex $C^m(t)$ in common with $A^i(t)$ and $A^j(t)$ by $A^{m_1}(t)$, $A^{m_2}(t), \cdots$ and the faces which have the vertex $C^n(t)$ in common with $A^i(t)$ and $A^j(t)$ by $A^{n_1}(t)$, $A^{n_2}(t), \cdots$. Let $A^{m_l}(t)$ be any one of the faces belonging to the former set. Let $\bar{A}^i(t)$, $\bar{A}^j(t)$ and $\bar{A}^{m_l}(t)$ be the three planes which contain the faces $A^i(t)$, $A^j(t)$ and $A^{m_l}(t)$ respectively. Further let $\tilde{A}^i(t)$, $\tilde{A}^j(t)$ and $\tilde{A}^{m_l}(t)$ be the closed negative half-spaces which are bounded by $\bar{A}^i(t)$, $\bar{A}^j(t)$ and $\bar{A}^{m_l}(t)$ respectively and contain $P(t)$. We denote the intersection of these half-spaces $\tilde{A}^i(t)$, $\tilde{A}^j(t)$ and $\tilde{A}^{m_l}(t)$ by $Q^{ijm_l}(t)$. That is, it is expressed as follows,

$$Q^{ijm_l}(t) = \tilde{A}^i(t) \cap \tilde{A}^j(t) \cap \tilde{A}^{m_l}(t).$$

Now taking the circular cone which is inscribed in the trihedral angle $Q^{ijm_l}(t)$, the part of the axis of the cone which is contained in $Q^{ijm_l}(t)$ is called the *axis of the trihedral angle* $Q^{ijm_l}(t)$ and the vertical angle of the cone is called *the vertical angle* of $Q^{ijm_l}(t)$. Let us denote the axis and the vertical angle of $Q^{ijm_l}(t)$ by $L^{ijm_l}(t)$ and θ_{ijm_l} respectively.

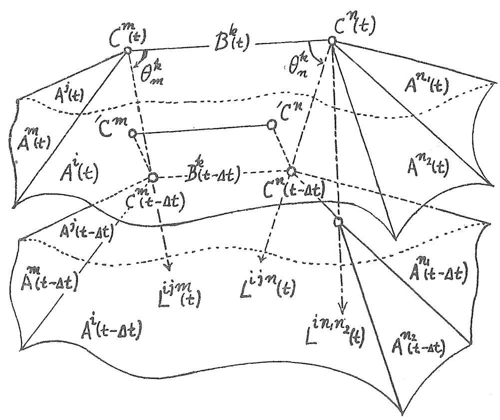

Fig. 3

Let $Q^{ijm_l}(t-\Delta t)$ be the interior parallel trihedral angle of $Q^{ijm_l}(t)$ at the distance Δt. Then the distance between the vertices $C^m(t)$ and $C^m(t-\Delta t)$ of the corresponding trihedral angles $Q^{ijm_l}(t)$ and $Q^{ijm_l}(t-\Delta t)$ is equal to $\Delta t \cdot \operatorname{cosec}(\theta_{ijm_l}/2)$. Hence it follows by our definition of the interior parallel polyhedra that, if Δt is a sufficiently small positive number and the parameter decreases from t to $t-\Delta t$, the extremity $C^m(t-\Delta t)$ of the edge $B^k(t-\Delta t)$ which corresponds to the one, $C^m(t)$, of the edge $B^k(t)$ moves along the axis of the trihedral angle whose vertical angle is the smallest one among the set $\{Q^{ijm_l}\}$ of the trihedral angles $Q^{ijm_l}(t)$; $l=1, 2, \cdots$, at the vertex $C^m(t)$ of $P(t)$. Taking one of such trihedral angles which have the smallest vertical angle, let it be expressed by $Q^{ijm}(t)$. Consequently let us denote the axis and the vertical angle of $Q^{ijm}(t)$ by $L^{ijm}(t)$ and θ_{ijm} respectively. In the same way, we denote the trihedral angle which has the smallest vertical angle among the set $\{Q^{ijn_l}(t)\}$ at the other extremity $C^n(t)$ of the edge $B^k(t)$ by $Q^{ijn}(t)$ and the axis and vertical angle of $Q^{ijn}(t)$ by $L^{ijn}(t)$ and θ_{ijn} respectively.

ii) The length of an edge

Let us denote the angle between the edge $B^k(t)$ and the axis $L^{ijm}(t)$ at the vertex $C^m(t)$ by θ^k_m and the corresponding angle at the vertex $C^n(t)$ by θ^k_n (see Fig. 3).

The angles θ^k_m and θ^k_n are called the *axial angles* to the edge $B^k(t)$. Then we have

Theorem 3. *If the sum of the axial angles* θ^k_m *and* θ^k_n *to the edge* $B^k(t)$ *of* $P(t)$ *is larger, smaller than or eqal to* π, *then the length of the edge* $B^k(t)$ *increases, decreases or is constant towards the interior of* $P(t)$ *in the neighbourhood of* t.

If $\Delta t\,(>0)$ is sufficiently small, two axes $L^{ijm}(t)$ and $L^{ijn}(t)$ are common to the parallel edges $B^k(t)$ and $B^k(t-\Delta t)$. Then since the quadrilateral $C^m(t)\,C^n(t)\,C^n(t-\Delta t)$ $C^m(t-\Delta t)$ is a trapezoid (see Fig. 3), it follows from $\theta^k_m+\theta^k_n \gtreqqless \pi$ that

$$l_k(t-\Delta t) \gtreqqless l_k(t).$$

Thus the theorem has been proved.

The theorem can be put in other forms which are more convenient. One of them is as follows. We drop the perpendiculars from $C^m(t-\Delta t)$ and $C^n(t-\Delta t)$ onto the face $A^i(t)$ of $P(t)$ and denote the feet by $'C^m$ and $'C^n$ (see Fig. 3). Then we have

$$\overline{'C^m\,'C^n} = \overline{C^m(t-\Delta t)C^n(t-\Delta t)} = l_k(t-\Delta t).$$

Since the quadrilateral $C^m(t)\ C^n(t)\ 'C^n\ 'C^m$ is a trapezoid it follows that

$$\angle\,'C^mC^m(t)C^n(t)+\angle\,C^m(t)C^n(t)'C^n \gtreqqless \pi \rightleftarrows l_k(t-\Delta t)\gtreqqless l_k(t). \tag{2}$$

Taking the trihedral angles $Q^{ijm}(t)$ and $Q^{ijn}(t)$, let α_{mi} and α_{mj} be the face angles of $Q^{ijm}(t)$ which have the edge $B^k(t)$ as their sides in common and α_m be the rest of the face angles of $Q^{ijm}(t)$. In the same way, α_{ni}, α_{nj} and α_n are the face angles of $Q^{ijn}(t)$. Then it follows that

$$\angle\,'C^mC^m(t)C^n(t) = \frac{1}{2}(\alpha_{mi}+\alpha_{mj}-\alpha_m),$$

$$\angle\,C^m(t)C^n(t)'C^n = \frac{1}{2}(\alpha_{ni}+\alpha_{nj}-\alpha_n).$$

Here we define α_{ijm} and α_{ijn} by

$$\alpha_{mi}+\alpha_{mj}+\alpha_m=2\alpha_{ijm}, \qquad \alpha_{ni}+\alpha_{nj}+\alpha_n=2\alpha_{ijn}.$$

Combining the result of the form (2) and these angles, we have

Corollary. *If* $2\alpha_{ijm}$ *and* $2\alpha_{ijn}$ *be the sums of face-angles of the trihedral angles* $Q^{ijm}(t)$ *and* $Q^{ijn}(t)$ *which define the axis* $L^{ijm}(t)$ *and* $L^{ijn}(t)$ *at the extremities of the edge* $B^k(t)$ *of* $P(t)$, *and* α_m *and* α_n *be the face-angles of* $Q^{ijm}(t)$ *and* $Q^{ijn}(t)$ *which have no bearing on the edge* $B^k(t)$, *then we have*

$$a_{ijm}+a_{ijn}-(a_m+a_n) \gtreqless \pi \rightleftarrows l_k(t-\Delta t) \gtreqless l_k(t).$$

iii) The negative edges and the vanishing of non negative-edges

It is easy to see that the area of the face $A^i(t)$ $(i=1,\cdots, f(t))$ of $P(t)$ does not increase when t decreases. On the other hand, the above result asserts that the length of an edge $B^k(t)$ of $P(t)$ may increase or decrease.

Here we put

Definition. *If the length of the edge $B^k(t)$ of $P(t)$ increases as the parameter t decreases, the edge $B^k(t)$ is called a negative edge in the interval in which the property holds. If a convex polyhedron $P(t)$ bears negative edges, $P(t)$ is called edge-singular; otherwise $P(t)$ is called edge-regular.*

Here it is possible that, when the decreasing parameter t converges to a critical value of the parameter, one or many faces of $P(t)$ which define the both extremities of an edge $B^k(t)$ vanish, and when the parameter becomes smaller than the critical value,

(i) *the negative edge $B^k(t)$ changes into a non negative edge,*

or

(ii) *the non negative edge $B^k(t)$ changes into a negative edge.*

Next let us estimate the critical value which corresponds to the vanishing of a non-negative edge. The theorem 3 shows that, if the edge $B^k(t)$ of $P(t)$ which is defined by the faces $A^i(t)$ and $A^j(t)$ is not a negative edge strictly then the axes $L^{ijm}(t)$ and $L^{ijn}(t)$ intersect each other.

Such a point of intersection is equidistant from the four faces $A^i(t)$, $A^j(t)$, $A^m(t)$ and $A^n(t)$ of the polyhedron $P(t)$. Let us denote the equal distance by $d(B^k(t))$. Then the sphere whose centre is the point of the intersection $L^{ijm}(t) \cap L^{ijn}(t)$ and whose radius is $d(B^k(t))$ touches the four faces $A^i(t)$, $A^j(t)$, $A^m(t)$ and $A^n(t)$. Let us define such a sphere as a *relative insphere* to the edge $B^k(t)$ of $P(t)$ and its radius $d(B^k(t))$ as a *relative inradius* of the non-negative edge $B^k(t)$.

If $t_0-d(B^k(t_0))>0$ and $t\rightarrow(t_0-d(B^k(t_0))+0$, the edge $B^k(t)$ of $P(t)$ converges to the centre of the relative insphere to the edge $B^k(t_0)$. If we denote the critical value which corresponds to the vanishing of the edge $B^k(t)$ by $\rho(B^k(t))$, then we have

Theorem 4. *If the relative insphere to the non negative edge $B^k(t)$ of an interior parallel polyhedron $P(t)$ is contained in $P(t)$, the value $t-d(B^k(t))$ is equal to the critical value $\rho(B^k(t))$ of the edge $B^k(t)$.*

1.4 The decomposition theorem II

i) Critical values

The transitions of the faces and the edges of the interior parallel polyhedron $P(t)$ which belongs to the sequence $\{P(t): 0 \leqq t \leqq r\}$ as the parameter t decreases from r to 0 have been definitely shown by the Theorems 2, 3 and 4.

Let us enumerate all the critical values of the parameter t which correspond to the vanishing of the faces $A^i(t)$; $i=1,\cdots,f$ and the edges $B^j(t)$; $j=1,\cdots,e(t)$ as follows :

$$\rho(A^i);\ i=1,\cdots,f: \qquad \rho(B^j(t));\ j=1,\cdots,e(t); \qquad 0\leqq t\leqq r.$$

Again let us put together these values of the two sets and arrange them in increasing order. We denote these numbers so-arranged as follows:

$$\rho_0,\ \rho_1,\ \cdots,\ \rho_N$$

where $\rho_0=0$ and $0\leqq N<\infty$.

These numbers are called the *critical values of the sequence* $\{P(t)\colon 0\leqq t\leqq r\}$ *of interior parallel polyhedra.* Any one of these critical values may correspond to the vanishing of one or more faces and edges.

We denote the semi open interval $(\rho_i, \rho_{i+1}]$ by I_i; $i=0,1,\cdots,N$ where $I_N=(\rho_N, r]$. Therefore, it follows that, *if* t, $t+\Delta t\in I_i$, *the parallel polyhedra* $P(t)$ *and* $P(t+\Delta t)$ *contain the same number of faces and edges and each of the faces and edges of* $P(t)$ *is parallel to the corresponding element of* $P(t+\Delta t)$.

ii) The second decomposition theorem

Theorem 5. *If* t, $t+\Delta t\in I_i$, *then*

$$P(t+\Delta t)=P(t)+\sum_{i=1}^{f(t)}\alpha^i(t,\Delta t)+\sum_{j=1}^{e(t)}\beta^j(t,\Delta t)+\sum_{k=1}^{v(t)}\gamma^k(t,\Delta t), \tag{3}$$

where $\alpha^i(t,\Delta t)$ *is a right prism with the base* $A^i(t)$ *and an altitude* Δt, $\beta^j(t,\Delta t)$ *a right prism with four edges which are parallel congruent with* $B^j(t)$ *and* $\gamma^k(t,\Delta t)$ *the compound pyramid with a pair of the opposite vertices* $C^k(t)$ *and* $C^k(t+\Delta t)$.

We can prove it by using Hadwiger's decomposition formula of the exterior parallel polyhedron ([7, p. 16]). Namely, if we denote the exterior parallel surface with the distance Δt of $P(t)$ by $'P(t+\Delta t)$, then we have

$$'P(t+\Delta t)=P(t)+\sum_{i=1}^{f(t)}{}'\alpha^i(t,\Delta t)+\sum_{j=1}^{e(t)}{}'\beta^j(t,\Delta t)+\sum_{k=1}^{v(t)}{}'\gamma^k(t,\Delta t) \tag{4}$$

where $'\alpha^i(t,\Delta t)$ is congruent to $\alpha^i(t,\Delta t)$; $'\beta^j(t,\Delta t)$ a cylindrical-sector whose altitude is equal to the length of the edge $B^j(t)$ and whose central angle is equal to the angle φ_j formed by the two positive normals to the faces which meet on the edge $B^j(t)$ of $P(t)$; $'\gamma^k(t,\Delta t)$ a spherical sector of the sphere with center $C^k(t)$ and with radius Δt whose solid angle is formed by the positive normals at $C^k(t)$ (see Fig. 4).

The $v(t)$ spherical sectors $'\gamma^k(t,\Delta t)$: $k=1,\cdots,v(t)$ are gathered up to a sphere whose radius is Δt by parallel displacement.

Then $\beta^j(t,\Delta t)$ is a circumscribed prism of $'\beta^j(t,\Delta t)$. However, if the edge $B^j(t)$ be a negative edge, then $l_j(t)>l_j(t+\Delta t)$. Hence $\beta^j(t,\Delta t)$ partly protrudes beyond $P(t+\Delta t)$ (see Fig. 6). Further, the compound pyramid $\gamma^k(t,\Delta t)$ is a figure circumscribed to the

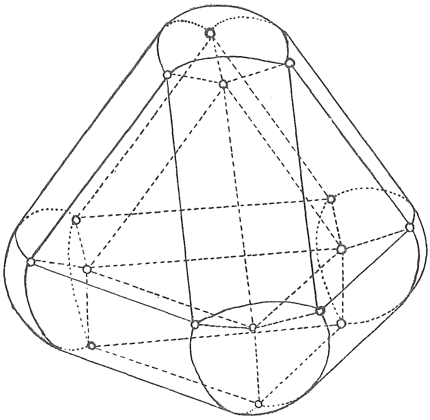

Fig. 4

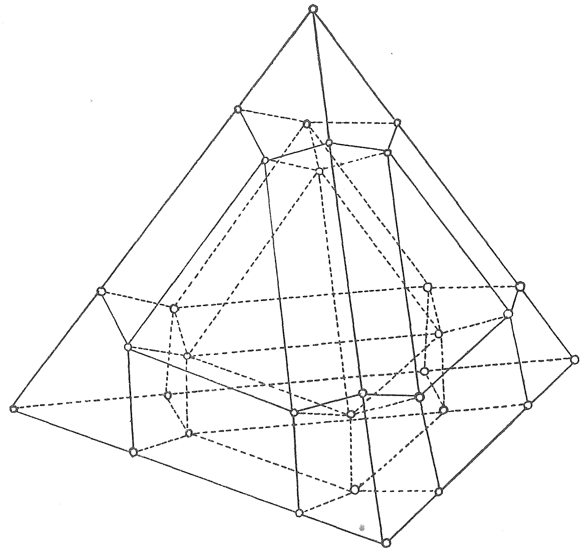

Fig. 5

spherical sector $'\gamma^k(t, \Delta t)$.

Now it should be noted that

(i) *By means of parallel displacement, we can gather up the* $v(t)$ *compound pyramids* $\gamma^k(t, \Delta t)$; $k=1, \cdots, v(t)$ *to a closed circumscribed polyhedron about a sphere whose radius is* Δt.

(ii) *If an edge* $B^j(t)$ *be a negative edge with the extremities* $C^m(t)$ *and* $C^n(t)$, *the terms of the decomposition formula* (3) *which make up for the excess of* $\beta^j(t, \Delta t)$ *from* $P(t+\Delta t)$ *should be nothing but* $\gamma^m(t,\Delta t)$ *and* $\gamma^n(t, \Delta t)$.

Then, due to the negativeness of the edge $B^j(t)$, *the polyhedron cicumscribed to a sphere of radivs* Δt *is not convex.* Further it is easy to see that the length of the edge of the circumscribed polyhedron which corresponds to an edge $B^j(t)$ is equal to $l_j(t+\Delta t)-l_j(t)$ ($\gtreqless 0$).

1.5. The characteristic function

i) The edge-curvature $M(t)$ and the quasi edge-curvature $M^*(t)$

Let the volume and the surface area of $P(t)$ be denoted by $V(t)$ and $S(t)$ respectively. The volume of the cylindrical-sector $'\beta^j(t, \Delta t)$ in the Hadwiger's decomposition formula (4) is expressed by $\frac{1}{2}\varphi_j\, l_j(t)^2$. The following expression was defined by W. Blaschke as the *edge-curvature* of $P(t)$, namely

$$M(t) = \frac{1}{2}\sum_{j=1}^{e(t)} l_j(t)\,\varphi_j, \tag{5}$$

where $l_j(t)$ is the length of the edge $B^j(t)$ and φ_j is the angle between the positive normals to the two faces of $P(t)$ which have the edge $B^j(t)$ in common.

Adopting Blaschke's notation, let us denote the vertex-curvature of $P(t)$ by $C(t)$. Denoting the volume of the exterior paralled surface $'P(t+\Delta t)$ of $P(t)$ by $'V(t+\Delta t)$, we can obtain the *Steiner's formulas* from the Hadwiger's decomposition formula (4) as follows:

$$(6) \qquad 'V(t+\Delta t) = V(t) + S(t)\Delta t + M(t)(\Delta t)^2 + \frac{1}{3}C(t)(\Delta t)^3$$

where $C(t)=4\pi$.

On the other hand, by reference to the definition of the edge-curvature $M(t)$ of $P(t)$, we define a *quasi edge-curvature* $M^*(t)$ of $P(t)$ by the form

$$(7) \qquad M^*(t) = \sum_{j=1}^{e(t)} l_j(t)\tan\frac{\varphi_j}{2}.$$

Next, in order to obtain the expression of the volume of the fourth term in the formula (3), we begin with the definition of the form-figure of $P(t)$.

ii) The definition of a form-figure

In the positive direction of the normal to the face $A^i(t)$ we draw the radius of the unit sphere whose centre is the origin and denote by $a^i(t)$ the extremity of the radius. The point $a^i(t)$ is called the *indicatrix point* of the face $A^i(t)$ and the unit sphere marked the $f(t)$ indicatrix points $a^i(t)$; $i=1,\cdots,f(t)$ is called the *normal indicatrix* of $P(t)$.

Through the indicatrix point $a^i(t)$, draw a plane, $\overline{\Omega}^i(t)$, parallel to the face $A^i(t)$. The plane $\overline{\Omega}^i(t)$ is called the *image plane of the face* $A^i(t)$.

Further, drawing the image planes $\overline{\Omega}^{i_1}(t),\cdots,\overline{\Omega}^{i_\mu}(t)$ of all the faces of $P(t)$ which possess an edge or a vertex of $P(t)$ in common with the face $A^i(t)$, we denote by $\Omega^i(t)$ the polygon on the image plane $\overline{\Omega}^i(t)$ which is formed by the μ lines of intersection of the plane $\overline{\Omega}^i(t)$ with the μ planes $\overline{\Omega}^{i_1},\cdots,\overline{\Omega}^{i_\mu}(t)$ respectively. Here it should be noted that the polygon $\Omega^i(t)$ is not always convex. We denote by $\Pi(t)$ the closed polyhedron which is enclosed by the $f(t)$ polygons $\Omega^1(t),\cdots,\Omega^{f(t)}(t)$. The polyhedron $\Pi(t)$ is called the *form-figure* of the closed convex polyhedron $P(t)$, and is expressed by

$$(8) \qquad \Pi(t) = \bigcup_{i=1}^{f(t)} \Omega^i(t).$$

It is easy to see that the form-figure $\Pi(t)$ of $P(t)$ is similar to the circumscribed polyhedron of the sphere with radius Δt which is composed of $v(t)$ compound pyramids $\gamma^k(t,\Delta t)$: $k=1,\cdots,v(t)$. *The ratio of the similitude* is $1:\Delta t$. Further it is easily seen that *the length of the image edge on the form-figure $\Pi(t)$ which corresponds to the edge $B^j(t)$ of $P(t)$ is equal to*

$$\frac{l_j(t+\Delta t)-l_j(t)}{\Delta t}.$$

If $B^j(t)$ is a negative edge of $P(t)$ and t, $t+\Delta t \in I_i$, then we have

$$l_j(t+\Delta t)-l_j(t)<0.$$

Hence, *the length of the image edge of $\Pi(t)$ which corresponds to the negative edge of $P(t)$ is negative and the parts of the form-figure $\Pi(t)$ which contain image edges corresponding to the negative edges of $P(t)$ are concave* (see Fig. 6).

In connection with the form-figure $\Pi(t)$ of a convex polyhedron $P(t)$, a new form is defined as follows:

$$\Pi^*(t) = \bigcap_{i=1}^{f(t)} \tilde{\Omega}^i(t), \tag{9}$$

where $\tilde{\Omega}^i(t)$ is a closed half-space which is bounded by the image plane $\bar{\Omega}^i(t)$ and contains the normal indicatrix to $P(t)$. By A. Dinghas [5] and H. Hadwiger [8], the form $\Pi^*(t)$ is called the form-figure of $P(t)$. But, in order to distinguish $\Pi^*(t)$ from $\Pi(t)$, let us call $\Pi^*(t)$ by the name the *quasi form-figure* of $P(t)$. *The quasi form-figure $\Pi^*(t)$ is convex. If $P(t)$ is edge-regular, $\Pi(t)$ is congruent to $\Pi^*(t)$. If $P(t)$ is edge-singular, $\Pi(t)$ contains the image edges corresponding to the negative edges of $P(t)$ but $\Pi^*(t)$ does not contain them. Instead of it, $\Pi^*(t)$ contains a set of edges which do not* correspond any edge of $P(t)$ (see Fig. 6).

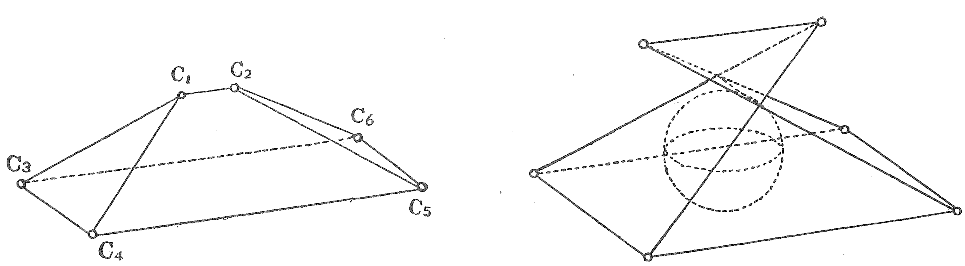

Fig. 6

Here, let us summarize the above result on the form-figure as follows:

Theorem 6. *If the convex polyhedron $P(t)$ is edge-singular, the form-figure $\Pi(t)$ of $P(t)$ is concave.*

iii) The characteristic function

The signed area of the closed polygon $\Omega^i(t)$ of the form-figure which is the image face of the face $A^i(t)$ of $P(t)$ is denoted by $\sigma_i(t)$. Then the *characteristic function* $\varkappa(t)$ of the convex polyhedron $P(t)$ is defined by

(10) $$\varkappa(t)=\sum_{i=1}^{f(t)}\sigma_i(t).$$

Here, taking an edge $B^j(t)$ of $P(t)$ which is defined by the faces $A^i(t)$ and $A^k(t)$ and whose extremities are the vertexes $C^m(t)$ and $C^n(t)$, let the image edge on the form-figure of the edge $B^j(t)$ of $P(t)$ be denoted by $b^j(t)$ and the image vertexes of the vertexes $C^m(t)$ and $C^n(t)$ by $c^m(t)$ and $c^n(t)$ respectively. If φ_i is the plane-angle of the dihedral angle of the edge $B^j(t)$, $l_j(t)$ the length of $B^j(t)$ and t, $t+\Delta t\in I_i$, then the signed area of the triangle $a^i(t)\,c^m(t)\,c^n(t)$ on the image face $\Omega^i(t)$ is given by

$$\frac{1}{2}\cdot\frac{l_j(t+\Delta t)-l_j(t)}{\Delta t}\tan\frac{\varphi_j}{2}$$

which is equal to the signed area of the triangle $a^j(t)c^m(t)c^n(t)$ on the image face $\Omega^j(t)$.

If the surface area $\varkappa(t)$ of the form-figure $\Pi(t)$ is measured with image edges $b^j(t)$; $j=1,\cdots,e(t)$ of $\Pi(t)$ as a guide instead of image faces $\Omega^i(t)$; $i=1,\cdots,f(t)$, it is given by

(11) $$\varkappa(t)=\sum_{j=1}^{e(t)}\frac{l_j(t+\Delta t)-l_j(t)}{\Delta t}\tan\frac{\varphi_j}{2}.$$

Next, we denote the surface area of the quasi form-figure $\Pi^*(t)$ by $\varkappa^*(t)$ and call it the ***quasi characteristic function*** or ***quasi characteristic*** of the convex polyhedron $P(t)$. When the parameter t decreases from r to 0, the quasi form-figure $\Pi^*(t)$ is taken away the image faces of the vanishing faces of $P(t)$ as often as the faces vanish. Hence, as the parameter t decreases from r to 0, the quasi characteristic function $\varkappa^*(t)$ is monotone increasing and is never less than 4π. Or it can be stated as follows:

Theorem 7. *The quasi characteristic function $\varkappa^*(t)$ of the sequence $\{P(t):0\leqq t\leqq r\}$ of the interior parallel surfaces of a convex closed polyhedron P is a monotone decreasing function and is never less than 4π.*

On the other hand, if the interior parallel polyhedron $P(t)$ is edge-singular and the concave parts of its form-figure $\Pi(t)$ enlarge, it is possible that the characteristic function $\varkappa(t)$ takes any negative value without restriction.

For example, let $C^1C^2-C^3C^4C^5C^6(t)$ be a pentahedron and φ_1 the plane angle of the dihedral angle along the edges $\overline{C^3C^4}(t)$ and $\overline{C^5C^6}(t)$ and φ_2 the one along the edges $\overline{C^3C^6}(t)$ and $\overline{C^4C^5}(t)$ (see Fig. 6). Then the characteristic function $\varkappa(t)$ is expressed as follows:

$$\varkappa(t)=2\tan^2\frac{\varphi_2}{2}\frac{\tan\varphi_2}{\tan\varphi_1}\left(\tan\frac{\varphi_2}{2}\tan\varphi_2-3\tan\frac{\varphi_1}{2}\tan\varphi_1\right).$$

Here, if $\varphi_1<\varphi_2$, the length of the edge $\overline{C^1C^2}(t)$ is getting shorter inward. If $\varphi_1=\varphi_2$, the length is constant. If $\varphi_1<\varphi_2$, the length becomes longer inward. For example, the value of $\varkappa(t)$ in case of $\varphi_1>\varphi_2$ is given as follows:

$$\varkappa(t)\Big|_{\substack{\varphi_1=150^\circ\\ \varphi_2=102^\circ}} = 15.97\cdots, \qquad \varkappa(t)\Big|_{\substack{\varphi_1=150^\circ\\ \varphi_2=100^\circ}} = -10.54\cdots,$$

Next, when $\varphi_1 < \varphi_2$, if s is a relative inradius of the edge $\overline{C^1C^2}(r)$ and $r > s$, the edge $\overline{C^1C^2}(t)$ of the pentahedron $P(t)$ vanished at the value $r-s$. Further when the parameter t decreases in succession beyond the critical value $r-s$, two faces $\overline{C^1C^3C^4}(t)$ and $\overline{C^2C^5C^6}(t)$ which have no points in common in the interval $(r-s, r)$ acquire a new negative edge in common and the edge $\overline{C^1C^2}(t)$ vanishes. As the result of the appearence of the negative edge, the form-figure $\Pi(t)$ changes to be concave in the interval $[0, r-s]$. Therefore the characteristic function $\varkappa(t)$ of the pentahedron $P(t)$ is not a monotone decreasing function in $0 \leqq t \leqq r$.

Further we can give various examples of a sequence $\{P(t) : 0 \leqq t \leqq r\}$ of the interior parallel polyhedra whose characteristic function $\varkappa(t)$ is not always a monotone function in the interval $0 \leqq t \leqq r$.

Here by $\{P\}$ we denote the set of all the convex polyhedra with interior points in the three dimensional Euclidean space E_3 and by $\varkappa(P)$ the characteristic function of P. Then we have

Theorem 8. *The functional $\varkappa(P)$ defined for the set $\{P\}$ takes any real number as its value and according as P is edge-regular or edge-singular, the value is either always greater or not always greater than 4π.*

Further we have

Theorem 9. *The characteristic function $\varkappa(t)$ of the sequence $\{P(t) : 0 \leqq t \leqq r\}$ of the interior parallel polyhedra of a convex polyhedron P is not always a monotone function.*

Here we can derive another theorem from the corollary of Theorem 3. Suppose that the edge $B^k(t)$ whose extremities are $C^m(t)$ and $C^n(t)$ is a negative edge of $P(t)$ and the faces $A^m(t)$ and $A^n(t)$ define the axes $L^{ijm}(t)$ and $L^{ijn}(t)$ to the edge $B^k(t)$ with faces $A^i(t)$ and $A^j(t)$ which meet on the edge $B^k(t)$. Let $a^i(t)$, $a^j(t)$, $a^m(t)$ and $a^n(t)$ be the indicatrix points of the faces $A^i(t)$, $A^j(t)$, $A^m(t)$ and $A^n(t)$ respectively. Then the trihedral angles $O-a^i(t)a^j(t)a^m(t)$ and $O-a^i(t)a^j(t)a^n(t)$ are the supplementary trihedral angles to the trihedral angles $Q^{ijm}(t)$ and $Q^{ijn}(t)$ respectively. Then applying the theorem that a face angle of a trihedral angle is a suplementary angle of the corresponding dihedral angle of the supplementary trihedral angle and the fact that the dihedral angle of the edge of the trihedral angle $O-a^i(t)$ $a^j(t)$ $a^m(t)$ is equal to the interior angle of the spherical triangle $a^i(t)a^j(t)a^m(t)$, we have

Theorem 10. *Let an edge $B^k(t)$ of the interior parallel polyhedron $P(t)$ of a convex polyhedron P be defined by the faces $A^i(t)$ and $A^j(t)$ and the axes $L^{ijm}(t)$ and $L^{ijn}(t)$ of the edge $B^k(t)$ be defined by the face $A^m(t)$ and $A^n(t)$. Let the interior angles of the spherical quadrilateral $a^i(t)a^n(t)a^j(t)a^m(t)$ which is constructed by the indicatrix points of the faces $A^i(t)$, $A^n(t)$, $A^j(t)$ and $A^m(t)$ be β_i, β_n, β_j, β_m. Then we have*

(12) $$\beta_m+\beta_n \gtreqqless \beta_i+\beta_j \rightleftarrows l_k(t) \gtreqqless l_k(t+\Delta t)$$

where t, $t+\Delta t \in I_i$, $i=1, 2, \cdots, N$ *and* $\Delta t>0$.

1.6. Differential formulas

i) Steiner's formulas

If we use the expressions $M^*(t)$ and $\varkappa(t)$ for the quasi edge-curvature and the characteristic function of $P(t)$, the second decomposition formula (3) can be expressed in comparison with the volumes $V(t)$ and $V(t+\Delta t)$ of $P(t)$ and $P(t+\Delta t)$ as follows:

$$V(t+\Delta t)=V(t)+S(t)\Delta t+M^*(t)(\Delta t)^2+\frac{1}{3}\varkappa(t)(\Delta t)^3.$$

This is Steiner's formula for the volume of the interior parallel polyhedron $P(t+\Delta t)$. It is easy to derive the corresponding formulas for the other quantities of $P(t+\Delta t)$ from the same formula (3) in comparison with the respective quantity in the same way. Then we have

Theorem 11. *If* t *and* $t+\Delta t$ *belong to the same interval* $I_i(i=0, 1, \cdots,$ *or* $N)$ *of the sequence* $\{P(t): 0\leqq t\leqq r\}$ *of the interior parallel polyhedra of a convex polyhedron* P, *the volume* $V(t+\Delta t)$, *the surface area* $S(t+\Delta t)$, *the quasi edge-curvature* $M^*(t+\Delta t)$ *and the edge-curvature* $M(t+\Delta t)$ *of* $P(t+\Delta t)$ *are expressed as follows*:

(13) $$V(t+\Delta t)=V(t)+S(t)\Delta t+M^*(t)(\Delta t)^2+\frac{1}{3}\varkappa(t)(\Delta t)^3,$$

(14) $$S(t+\Delta t)=S(t)+2M^*(t)\Delta t+\varkappa(t)(\Delta t)^2,$$

(15) $$M^*(t+\Delta t)=M^*(t)+\varkappa(t)\Delta t,$$

(16) $$M(t+\Delta t)=M(t)+\mu(t)\Delta t$$

where $\mu(t)$ *is the edge-curvature of the form-figure* $\Pi(t)$.

ii) Differential formulas

First, since the surface area $S(t)$ of the interior parallel polyhedra $P(t)$ is continuous over the interval $0\leqq t\leqq r$, we readily obtain from the formula (13) the following formula

(17) $$V'(t)=S(t)$$

where the prime means the differentiation with respect to t.

Next, in order to obtain the differential formulas for the rest of the quantities, we should keep in mind the continuity of the quantities, that is to say, $M^*(t)$, $\varkappa(t)$ and $\mu(t)$. In the first place, we have

Theorem 12. *The quasi edge-curvature $M^*(t)$ of the interior parallel polyhedron $P(t)$ of a convex polyhedron P with insphere of radius r (>0) is, in general, discontinuous at the critical values $\rho_i(i=1,\cdots, N)$. That is, we have*

$$M^*(\rho_i+0)\leqq M^*(\rho_i-0)$$

where the equality holds only when the limiting figures of all the vanishing faces as $t\to\rho_i+0$ are limited to the vertices alone.

For the sake of brevity, we omit the proof. For the detail see [10, p. 137] or [11, pp. 21–22].

The discontinuity of the characteristic function $\varkappa(t)$ is definitely shown by Theorem 12 and the fact that *the characteristic function $\varkappa(t)$ is nothing less than the quasi edge-curvature to the form-figure $\Pi(t)$.*

Further, since the form-figure $\Pi(t)$ changes its shape at each critical value ρ_i ; $i=1,\cdots$, N, it is clear that the edge-curvature $\mu(t)$ of $\Pi(t)$ is discontinuous at each critical value. In connection with Theorem 9, it is easy to see that $\mu(t)$ is not always a monotone function. Hence we have

Theorem 13. *The characteristic function $\varkappa(t)$ and the edge-curvature $\mu(t)$ of the form-figure $\Pi(t)$ of the interior parallel sequence $\{P(t):0\leqq t\leqq r\}$ are discontinuous at the critical values ρ_i : $i=1,\cdots$, N and are not always monotone functions.*

Then, the differential formulas of $S(t)$, $M^*(t)$ and $M(t)$ are readily obtained from (14), (15) and (16) respectively. Taking account of Theorems 12 and 13 the results are summarized in the following theorem:

Theorem 14. *If $V(t)$, $S(t)$, $M^*(t)$ and $M(t)$ are the volume, the surface area, the quasi edge-curvature and the edge-curvature of the interior parallel polyhedron $P(t)$ of a convex polyhedron P with inradius r (>0), then $V(t)$ is differentiable in $0\leqq t\leqq r$ and $S(t)$, $M^*(t)$, $M(t)$ are differentiable in (ρ_i, ρ_{i+1}) ; $i=0,\cdots,N$ and one-sided differentiable at the critical values ρ_i; $i=0,\cdots$, N.*

That is, we have in $0\leqq t\leqq r$

$$V'(t) = S(t) \tag{17}$$

and in (ρ_i, ρ_{i+1}); $i=0,\cdots$, N

$$S'(t) = 2M^*(t), \tag{18}$$

$$M^{*\prime}(t) = \varkappa(t), \tag{19}$$

$$M'(t) = \mu(t). \tag{20}$$

And we have at a critical value ρ_i : $i=1, 2,\cdots$, or N

(21) $$'S(\rho_i) = 2M^*(\rho_i - 0) \geqq S'(\rho_i) = 2M^*(\rho_i + 0),$$

(22) $$'M^*(\rho_i) = \varkappa(\rho_i - 0) \gtrless M^{*\prime}(\rho_i) = \varkappa(\rho_i + 0),$$

(23) $$'M(\rho_i) = \mu(\rho_i - 0) \gtrless M'(\rho_i) = \mu(\rho_i + 0)$$

where the equality in (21) *holds only when $P(t)$ contains no faces which converge to edges of $P(\rho_i)$ as $t \to \rho_i + 0$.*

§ 2. Approximations to a closed convex surface and formulas of differentiation and integration

2.1. The approximation theorem I

i) In this section and in all succeeding sections we shall use the phrase "*convex body*" to mean "*closed bounded convex set with interior points*", and "*closed convex surface*" to mean "*boundary of a convex body*".

Theorem 15. *If K is a convex closed surface and if ε is a given positive number, there are convex polyhedra P, P_ε such that*

$$P \subset K \subset P_\varepsilon$$

and P is an interior parallel polyhedron of P_ε at the distance ε.

Proof. K can be covered by a finite number of closed spheres with radius ε whose centres are contained in K. The convex closure of centres of such spheres is a convex polyhedron. We denote it by P. Further, taking a polyhedron to which P is an interior parallel polyhedron at the distance ε, we denote it by P_ε. Then it is clear that $P \subset K \subset P_\varepsilon$ and the theorem has been proved.

ii) Extreme points and extreme supporting planes

Let us denote the coordinates of a point X by (x_1, x_2, x_3), the closed segment joining the points X, Y by $\overline{XY}$ and the inner product $x_1 y_1 + x_2 y_2 + x_3 y_3$ of two points X, Y by $X \cdot Y$. For convenience, let us denote the equation of a plane which does not pass through the origine 0 by the form $A \cdot X = 1$, that is:

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = 1.$$

Definition. (i) *A point X is said to be an extreme point of the convex body K if $X \in K$ and there are no two points X_1, X_2 of K such that*

$$X \in \overline{X_1 X_2} \qquad (X \neq X_1,\ X \neq X_2)$$

ii) *A plane Π, $A\cdot X=1$ is said to be an extreme supporting plane of a closed convex surface K if it does not cut K and it is not possible to find two planes Π_1, Π_2, say $A_1\cdot X=1$ and $A_2\cdot X=1$, which do not cut K and are such that $A=\lambda A_1+\mu A_2$, $\lambda+\mu=1$, $\lambda>0$, $\mu>0$.*

Then we have the following two theorems (see [6, p. 24 and p. 27])

Theorem (1). *A convex body is the closure of the convex cover of its extreme points.*

Theorem (2). *A closed convex body is the intersection of the closed half spaces bounded by it sextreme supporting planes.*

2.2. The approximation theorem II

i) Lemma

Lemma. *If K is a convex body and a point X belongs to K, X is an extreme point of K, or an interior point of* 1*-simplex,* 2*-simplex or* 3*-simplex whose vertices are extreme points of K.*

Proof. X is a frontier point, or an interior point of K.

First let us suppose that X is a frontier point. Then by the quoted theorem (2), through the point X there passes at least one extreme supporting plane. Let Π be one of them. Then the supporting set $\Pi\cap K$ is a point, a line-segment or a two dimensional convex set with interior points on Π. If it is a point, the point $\Pi\cap K$ is none other than X. That is, X is an extreme point of K. If it is a line-segment, X is its extremity or its interior point. In the former case X is an extreme point and in the latter case it is an interior point of 1-simplex whose vertices are the extreme points of K. Next, if the set $\Pi\cap K$ is a convex set with interior points on Π, the point X is a frontier point or an interior point of the set $\Pi\cap K$. Here, let the boundary curve of the set $\Pi\cap K$ denote by U. If X is on the curve U, then it is clear that X is an extreme point or an interior point of 1-simplex whose extremities are extreme points of K. Next suppose that X is an interior point of the set $\Pi\cap K$ which is enclosed by the closed convex curve U. Then there are at least three extreme points on U. Let one of them be denoted by X^1. Then the straight line joining X and X^1 intersects U at a point other than X^1, say X^2. Then the point X^2 is an extreme point or an interior point of 1-simplex whose extremities are extreme points of K, say X^3 and X^4. Then in the former case X is an interior point of 1-simplex whose vertices are the extreme points X^1 and X^2 of K and in the latter case X is an interior point of 2-simplex whose vertices are the extreme points X^1, X^3 and X^4 of K.

Finally suppose that X is an interior point of K. By virtue of the quoted theorem (1) there are at least four extreme points on the surface of K which are not contained in a plane. Let four points of such a set be denoted by X^i; $i=1,\cdots,4$. Then X is an interior point, on a face, on an edge or an exterior point of the 3-simplex $X^1X^2X^3X^4$. If

X is an interior point, on a face or an edge of the 3–simplex, X is an interior point of 3–, 2– or 1–simplex whose vertices are extreme points of K. If X is an exterior point of the simplex, take one of the vertices of the simplex, say X^1, and let us draw a straight line joining the two points X and X^1. Then the line intersects the surface of K at a point other than the point X^1, say X^5. Then X^5 is an extreme point of K or an interior point of 1– or 2–simplex whose vertices are extreme points of K. Hence X is an interior point of 1–, 2– or 3–simplex whose vertices are extreme points of K. Thus the theorem has been proved.

ii) The approximation theorem II

Theorem 16. *If K is a convex body with the origin O as its interior point and if ε is a given positive number, there exists a convex polyhedron Q all of whose vertices are the extreme points of K such that*

$$Q \subset K \subset (1+\varepsilon)Q$$

where $(1+\varepsilon)Q$ is a polyhedron similar to Q whose centre of similitude is O.

Proof. Let h be the minimum value of the supporting function. There is a positive number ε_0 such that $0 < \varepsilon_0 < h\varepsilon$. Then by the approximation theorem 15 we can choose a convex polyhedron P such that

$$P_{\varepsilon_0} \subset K \subset P,$$

where P_{ε_0} is an interior parallel polyhedron to P at the distance ε_0. Here, taking all the vertices of P_{ε_0} which are not extreme points of K, we can let each of them correspond to a set of the extreme points of K by the lemma such that

i) the extreme points are the vertices of 1–, 2– or 3–simplex,

and

ii) such a simplex contains the vertex in question of P_{ε_0} as its interior point.

Now, leaving the vertices of P_{ε_0} which are the extreme points of K as it is, let each of the vertices of P_{ε_0} which is not an extreme point of K be replaced by the vertices of the simplexes of the above defined set. Let the polyhedron so obtained be denoted by P'. Further let us denote by Q the closure of the convex cover of the polyhedron P'. Then we have

$$P_{\varepsilon_0} \subset Q \subset K \subset P.$$

On the other hand, since $h\varepsilon > \varepsilon_0$, we have $P \subset (1+\varepsilon)Q$, that is to say,

$$Q \subset K \subset (1+\varepsilon)Q.$$

Thus the theorem has been proved.

2.3. The approximation theorem III

i) Polar duality

In order to obtain the approximation theorem III, we shall use the concept of polar duality.

Definition. *If K is any subset of the 3-dimensional Euclidean space E_3, then the polar of K is such a subset K^* defined by*

$$K^* = \{ Y : X \cdot Y \leqq 1, \ X \in K \}$$

where X and Y are (x_1, x_2, x_3) and (y_1, y_2, y_3) and $X \cdot Y = x_1y_1 + x_2y_2 + x_3y_3$.

The following properties of the polar duality are well-known.

(a) *If K is convex, K^* is convex.*

(b) *If $K_1 \subset K_2$, then $K_1^* \supset K_2^*$.*

(c) *The polar of an extreme point X of a convex body K is an extreme supporting plane of the polar K^*.*

(d) *If K^* is the polar of K, then the polar of λK is $\frac{1}{\lambda} K^*$.*

Then we have

Theorem 17. *If K is a convex body with the origin O as an interior point and if ε is a given positive number, there exists a convex polyhedron P each of whose faces is on an extreme supporting plane of K respectively and such that*

$$K \subset P \subset (1+\varepsilon) K. \tag{24}$$

Proof. Let K^* be the polar of K. Then we can suppose from the approximation theorem II that there exists a convex polyhedron P^* such that

$$\frac{1}{1+\varepsilon} K^* \subset P^* \subset K^*$$

and all the vertices of P^* are extreme points of K^*.

Now, if the polar of P^* is denoted by P, then we have from the above properties (a) and (b) that

$$K \subset P \subset (1+\varepsilon) K,$$

and from the property (c) that each of the faces of P is on one of the extreme supporting planes of K respectively. Thus the theorem has been proved.

2.4. Quantities associated with a convex body

Let $\mathfrak{A}$ be the class of convex bodies in a closed bounded portion of E_3. As is well-known, by using the definition of a neighbourhood $\mathfrak{U}(K, \delta)$ of a convex body K, we can define a metric in $\mathfrak{A}$ as follows.

Let K_1 and K_2 be two members of $\mathfrak{A}$, and let δ_1 be the lower bound of positive numbers δ such that $\mathfrak{U}(K_1, \delta) \supset K_2$ and δ_2 be the lower bound of positive numbers δ such that $\mathfrak{U}(K_2, \delta) \supset K_1$. Then the distance between K_1 and K_2 is defined to be

$$\Delta(K_1, K_2) = \delta_1 + \delta_2 . \tag{25}$$

Here, let $\{\varepsilon_i\}$ be a series of positive numbers $\varepsilon_i : i=1, 2, \cdots$ such that ε_i tends to zero as $i \to \infty$. Next, let P^i be a convex polyhedron each face of which is on an extreme supporting plane of K respectively and such that

$$K \subset P^i \subset (1+\varepsilon_i) K.$$

Moreover, let us consider that P^{i+1} contains a set of faces in common with all faces of P^i and

$$P^i \supset P^{i+1} \qquad i=1, 2, \cdots.$$

Then it is easy to see that

$$\Delta(K, P^i) \geqq \Delta(K, P^{i+1}) \qquad i=1, 2, \cdots \tag{26}$$

and

$$\lim_{i \to \infty} \Delta(K, P^i) = 0. \tag{27}$$

Now, let us denote the volume, surface area, quasi edge-curvature and edge-curvature of the polyhedron P^i by $V(P^i)$, $S(P^i)$, $M^*(P^i)$ and $M(P^i)$ respectively.

Then we can prove that *each sequence of these functionals, that is* $\{V(P^i)\}$, $\{S(P^i)\}$, $\{M^*(P^i)\}$ and $\{M(P^i)\}$ *is a monotone decreasing sequence*[2]. In the present case, we can prove the monotone decreasing property of the sequence $\{M^*(P^i)\}$ by showing that the quasi edge-curvature of the lateral edges of a pyramid is larger than that of the edges of its base. It is well-known that the same property is possessed by the other functionals (for example, see [7]).

Here let us summarize the above-obtained results.

(i) *The given convex body K is the intersection of the closed half-spaces bounded by its extreme supporting planes.*

(ii) *The given convex body K can be approximated with a monotone decreasing sequence* $\{\varepsilon_i\}$ *of given positive numbers* $\varepsilon_1 > \varepsilon_2 > \cdots$ *and by the sequence* $\{P^i\}$ *of polyhedra* P^i $(i=1, 2, \cdots)$ *each of whose faces is on an extreme supporting plane of K as follows*:

$$P^1 \supset P^2 \supset \cdots \supset K.$$

---

2) Here it should be noted that $\{M^*(P^i)\}$ is a monotone decreasing functional. Because, in general even if $P \subset Q$, it does not always hold that $M^*(P) < M^*(Q)$, that is to say, it may be that $M^*(P) > M^*(Q)$.

(iii) *Each of the sequences* $\{V(P^i)\}$, $\{S(P^i)\}$, $\{M^*(P^i)\}$ *and* $\{M(P^i)\}$ *of the quantities* $V(P^i)$, $S(P^i)$ $M^*(P^i)$ *and* $M(P^i)$; $i=1, 2, \cdots$ *is a monotone decreasing functional respectively, that is*:

$$V(P^1) \geqq V(P^2) \geqq \cdots, \qquad S(P^1) \geqq S(P^2) \geqq \cdots,$$

$$M^*(P^1) \geqq M^*(P^2) \geqq \cdots, \qquad M(P^1) \geqq M(P^2) \geqq \cdots.$$

Therefore we can conclude that there exist the infimums of these quantities, that is to say, $\inf V(P^i)$, $\inf S(P^i)$, $\inf M^*(P^i)$ and $\inf M(P)$. Moreover it is easy to show the uniquencess of each infimum of these quantities.

Hereupon *let us define the volume, surface area, total mean curvature and integral of mean curvature of the given convex body* K *by the* $\inf V(P^i)$, $\inf S(P^i)$, $\inf M^*(P^i)$ *and* $\inf M(P^i)$ *respectively. We denote them by* V, S, M^* *and* M *respectively.*

2.5. The sequence of interior parallel surfaces of a closed convex surface

i) Definitions

Let K be a convex body with interior points and $r\,(>0)$ the radius of its insphere. If the set of all the extreme supporting planes of K is denoted by $\{\bar{A}\}$, K is expressed from the quoted theorem (2)

$$K = \cap \tilde{A}$$

where $\tilde{A}$ is a closed negative half-space whose boundary is the extreme supporting plane $\bar{A}$.

Now we denote by $\bar{A}(t)$ the plane parallel to a plane $\bar{A}$ whose positive normal is parallel to that of $\bar{A}$ and whose distance from $\bar{A}$ is equal to $r-t$ inwardly along the negative normal of $\bar{A}$ and by $\tilde{A}(t)$ the closed negative half-space which is bounded by $\bar{A}(t)$. Then we define an interior parallel surface $K(t)$ to K at the distance $r-s$ by the form

$$K(t) = \cap \tilde{A}(t).$$

Here a parallel plane $\bar{A}(t)$ to the supporting plane $\bar{A}$ of K is not always a supporting plane of $K(t)$ and if $t>r$, $K(t)$ is a null set. Hence we can define the interior parallel surface $K(t)$ of K in the interval $0<t\leqq r$. We define *the kernel* of them as the limiting figure of $K(t)$ as $t\to 0$ and denote it by $K(0)$. Including the kernel $K(0)$, thus we define the *sequence of interior parallel surface of a closed convex surface with inradius* $r(>0)$ and denote it by $\{K(t): 0\leqq t\leqq r\}$.

Taking an interior parallel surface $K(t)$ of K and its supporting plane $\bar{A}(t)$, we call the intersection $\bar{A}(T) \cap K(t)$ the *supporting set* of $K(t)$ on the plane $\bar{A}(t)$ and denote it by $A(t)$. Then it is easily seen that *a supporting set* $A(t)$ *is a point, a line-segment of finite length or a convex flat-part with interior points on the supporting pla ne* $\bar{A}(t)$.

Further we can define the *indicatrix point* of a supporting set $A(t)$ and the *normal indicatrix* of $K(t)$ as the limiting figure of the corresponding figures of the approximating polyhedra to the convex body $K(t)$. Here, *if all the points of the surface* $K(t)$ *are regular, then the normal indicatrix of* $K(t)$ *is the unit-sphere itself. A singular point of the surface* $K(t)$ *corresponds to a blank portion of the normal indicatrix.*

Next let us define the form-figure $\Pi(t)$ of the surface $K(t)$. First, taking an extreme supporting plane $\bar{A}(t)$ of $K(t)$, let $\bar{\Omega}(t)$ be a tangent plane to the normal indicatrix of $K(t)$ which is parallel to $\bar{A}(t)$ and passes through the indicatrix point $a(t)$ of $\bar{A}(t)$. And we call the plane $\bar{\Omega}(t)$ the image plane of the plane $\bar{A}(t)$. Next let $\{\bar{A}'(t)\}$ be the class of extreme supporting planes which have the frontier points of $K(t)$ in common with $\bar{A}(t)$ and let $\{\bar{\Omega}'(t)\}$ be the class of the image planes of the class $\{\bar{A}'(t)\}$. Let $\Omega(t)$ be the closed convex curve on the plane $\bar{\Omega}(t)$ which is enclosed by the class $\{\bar{\Omega}'(t)\cap\bar{\Omega}(t)\}$ of the intersections $\bar{\Omega}'(t)\cap\bar{\Omega}(t)$. Then the *form-fifigure* $\Pi(t)$ of the surface $K(t)$ is defined by

$$\Pi(t) = \bigcup \Omega(t).$$

In connection with the definition of the form-figure, let $\tilde{\Omega}(t)$ be a closed negative half-space which is bounded by the image plane $\bar{\Omega}(t)$ of $\bar{A}(t)$ and which contains the normal indicatrix of $K(t)$. Then we define the *quasi form-figure* $\Pi^*(t)$ by the form

$$\Pi^*(t) = \bigcap \tilde{\Omega}(t).$$

It is clear that *the qnasi form-figure* $\Pi^*(t)$ *is always convex but the form-figure* $\Pi(t)$ *is not always convex.*

In conclusion we should define the *characteristic function* $\varkappa(t)$ of $K(t)$ as the signed surface area of the form-figure $\Pi(t)$ of $K(t)$ and the *quasi chracteristic function* $\varkappa^*(t)$ of $K(t)$ as the surface area of the quasi form-figure $\Pi^*(t)$ of $K(t)$. Here $\varkappa(t)$ *and* $\varkappa^*(t)$ *are none other than the total mean curvatures of* $\Pi(t)$ *and* $\Pi^*(t)$ *respectively.* At this time we define the integral of mean curvature of $\Pi(t)$ as *secondary characteristic* and denote it by $\mu(t)$.

ii) The properties of the characteristic function $\varkappa(t)$

Definition. (i) *A frontier point of a convex body is regular if it lies on only one supporting plane.*

(ii) *A supporting plane of a convex body is regular if it meets the body in only one point.*

Then it is easy to see that

(1) *the set of regular frontier points of a convex body is dense in the frontier of the body, and*

(2) *if every frontier point of* $K(t)$ *is regular then the frontier is mapped onto the*

normal indicatrix by the mapping by parallel supporting planes and the characteristic function $\varkappa(t)$ and the quasi-characteristic $\varkappa^*(t)$ are equal to 4π.

Further, it is clear that *a supporting plane* $\bar{A}$ *at a regular frontier point of a convex body* K *is an extreme supporting plane of* K.

If $\bar{A}(t)$ is a suppoorting plane at a regular frontier point of a convex body $K(t)$, the supporting set $\bar{A}(t) \cap K(t) = A(t)$ is called to be *regular.* Further the locus of the regular supporting set $A(t)$ of the sequence $\{K(t) : 0 \leqq t \leqq r\}$ is called an *inner body to the regular supporting set* A of the convex body K and it is denoted by $I(A)$.

Here in the same manner as in the case of a convex polyhedron it is proved by the Blaschke selection theorem that *there is such a class of spheres as touch the regular supporting set* $A(t)$ *of* $K(t)$ *and are contained in* $K(t)$ *and have the maximum radius.* Then such a sphere is called the *relative insphere* to the regular supporting set $A(t)$ and the radius of the relative insphere the *relative inradius* to the set $A(t)$.

Then corresponding to the Theorem 2 in the case of the interior parallel polyhedra, we have

Theorem 18. *If* $d(A)$ *is the relative inradius to a regular supporting set* A *of a closed convex surface* K, *then the regular supporting set* $A(t)$ *of the interior parallel surface* $K(t)$ *vanishes at the value* $r-d(A)$ $(\geqq 0)$.

Here, the surface normal to a regular supporting set $A(t)$ of $K(t)$ may displace continuously along the surface $K(t)$ and the relative inradius $d(A(t))$ may be a continuous function with respect to the direction of the surface normal. That is to say, in case of a convex body K, the value $r-d(A)$ may be continuous. Hence it follows that *the characteristic function* $\varkappa(t)$ *of the sequence* $\{K(t);\ 0 \leqq t \leqq r\}$ *is not always a step-function and in general may be a continuous function of the parameter* t.

However, the vanishing of the following portions of the surface $K(t)$ gives rise to discontinuities of the characteristic function $\varkappa(t)$ of $K(t)$, that is to say;

(1) *a flat-part of the surface* $K(t)$ *whose normal direction is discontinuous to the neighbouring ones, and*

(2) *a rectilinear edge.*

Therefore, the values of the parameter t corresponding to the discontinuous points of the characteristic function $\varkappa(t)$ are called the *critical values* of the parameter of the sequence $\{K(t) : 0 \leqq t \leqq r\}$.

Now, *the numbers of the flat-parts and the rectilinear edges of a closed convex surface are both enumerable.* Therefore *the number of the critical valves of the characteristic function* $\varkappa(t)$ *is at most enumerable.* Moreover, if we use the definitions of the terms *negative edge, edge-singular* and *edge- regular* just as they were given for the case of polyhedra, we have in general

Theorem 19. *The characcteristic function* $\varkappa(t)$ *of the sequence* $\{K(t) : 0 \leqq t \leqq r\}$ *of interior parallel surfaces of a closed convex surface* K *with inradius* $r\,(>0)$ *takes any real number and is not always a monotone function but the quasi characteristic function* $\varkappa^*(t)$ *is a monotone decreasing function. And we have*

(28) $$\varkappa(t) \leqq \varkappa^*(t), \qquad 4\pi \leqq \varkappa^*(t), \qquad 0 \leqq t \leqq r$$

where the equality in the former holds when $K(t)$ is edge-regular.

2.6. Differential formulas and integral formulas

i) Quantities and Steiner's formulas

On account of §§ 2.3 and 2.4, we can suppose a set of the functions $V(t)$, $S(t)$, $M^*(t)$, $M(t)$, $\varkappa(t)$, $\varkappa^*(t)$ and $\mu(t)$ as the enclosed volume, surface-area, total mean curvature, integral of mean curvature, characteristic function, quasi characteristic and secondary characteristic of a convex body $K(t)$ of the interior parallel sequence $\{K(t): 0 \leqq t \leqq r\}$.

In general, *the characteristic function $\varkappa(t)$ is not a step-function.* Hence *the extended Steiner's formulas of the volume, surface area, quasi edge-curvature and edge-curvature in case of a convex polyhedron cannot be generalized for the present case.*

ii) Differential formulas

But the differential formulas (17), (18), (19) and (20) can be generalized for a sequence $\{K(t): 0 \leqq t \leqq r\}$ of interior parallel surfaces of a closed convex surface K with inradius $r(>0)$. We can obtain them from the following theorem (see [12, P. 353]).

Theorem. *If a sequence of convex functions converges to a limit function, the derivatives of the functions of the sequence converge to the derivative of the limit function, provided that the latter exists, that is, with the exception at most of an enumerable set of points.*

Hence we have on account of the approximation theorem III

Theorem 20. *The enclosed volnme $V(t)$ of the interior parallel surface $K(t)$ of a convex closed surface K with inradius $r(>0)$ is differentiable in $0 < t \leqq r$, that is, we have*

(29) $$V'(t) = S(t).$$

The surface area $S(t)$, total mean curvature $M^*(t)$ and integral of mean curvature $M(t)$ are differentiable at every value t except for the critical values and we have

(30) $$S'(t) = 2M^*(t),$$

(31) $$M^{*\prime}(t) = \varkappa(t),$$

(32) $$M'(t) = \mu(t).$$

At the critical value ρ, $S(t)$, $M^*(t)$ and $M(t)$ are one-sided differentiable and we have

(33) $$'S(\rho) = 2M^*(\rho-0) \geqq S'(\rho) = 2M^*(\rho+0),$$

(34) $$'M^*(\rho) = \varkappa(\rho-0) \gtreqless M'^*(\rho) = \varkappa(\rho+0),$$

$$'M(\rho) = \mu(\rho-0) \gtrless M'(\rho) = \mu(\rho+0). \tag{35}$$

iii) Integral formulas

Since the surface area $S(t)$ is continuous in $[0, r]$, we get the integral formula for the volume $V(t)$ of an interior parallel surface $K(t)$ of a closed convex surface K with inradius $r(>0)$ from the differential formula (29) as

$$V(t) = \int_0^t S(t)\,dt$$

where $0 \leqq t \leqq r$.

Against this, however the total mean curvature $M^*(t)$, characteristic function $\varkappa(t)$ and secondary characteristic $\mu(t)$ are discontinuous at the critical values and the number of the critical values is at most enumerable. Hence the surface area $S(t)$, total mean curvature $M^*(t)$ and integral of mean curvature $M(t)$ are integrable. Thus we have

Theorem 21. *The volume $V(t)$, surface-area $S(t)$, total mean-curvature $M^*(t)$ and integral of mean-curvature $M(t)$ of an interior parallel surface $K(t)$ of a closed convex surface K with inradius $r(>0)$ are integrable in $[0, t]$ and we have*

$$V(t) = \int_0^t S(s)ds, \tag{36}$$

$$S(t) = \int_0^t 2M^*(s)ds + S(0), \tag{37}$$

$$M^*(t) = \int_0^t \varkappa(s)sd + M^*(0), \tag{38}$$

$$M(t) = \int_0^t \mu(s)ds + M(0) \tag{39}$$

where, if K is a surface with plane-kernel Q of the plane area F and the boundary-length U, then we have

$$S(0) = 2F, \qquad M^*(0) = \int_Q \varkappa_0(u)\,du, \qquad M(0) = \frac{\pi}{2}U;$$

if K is a surface with line-kernel of length L, then we have

$$S(0) = 0, \qquad M^*(0) = \frac{1}{2}\varkappa_0 L, \qquad M(0) = \pi L,$$

and if K is a surface with point-kernel, then we have

$$S(0) = M^*(0) = M(0) = 0\text{[3]}.$$

3) See for the detail [9, pp. 43-44].

iv) The convexity of Quantities

Suppose that the parameter t of the sequence $\{K(t): 0 \leqq t \leqq r\}$ decreases from r to zero. If the vanishings of extreme supporting planes at a critical value ρ bring some of new negative edges of $K(t)$, the form-figure $\Pi(t)$ adds some of concave parts itself. Then the characteristic function $\varkappa(t)$ decreases its value by the amount equal to the the new added negative area. But the vanishing of extreme supporting planes gives rise to the enlargement of the form-figure $\Pi(t)$ except for the addition of the negative edges, so that the values $\varkappa(t)$ and $\mu(t)$ increase. Hence if we suppose the case in which parameter t increases from 0 to r, we have

Theorem 22. *Except for the critical values of the parameter of the sequence* $\{K(t): 0 \leqq t \leqq r\}$ *of the interior parallel surfaces of a closed convex surface* K *with inradius* r (>0), *we have*

$$(40) \qquad \varkappa'(t) = \varkappa^{*\prime}(t) \leqq 0, \qquad \mu'(t) \leqq 0.$$

On the other hand, by the differential formulas (29), (30), (31) and (32), and the results that $M^*(t) \geqq 0$, $\varkappa(t) \gtreqqless 0$, and (40), the following theorem can be definitely shown.

Theorem 23. *The volume* $V(t)$ *of a interior parallel surface* $K(t)$ *of a closed convex surface* K *with inradius* r $(>o)$ *is a convex function, but its surface area* $S(t)$ *may be a convex function or a concave function. The total mean curvature* $M^*(t)$ *and the integral of mean curvature* $M(t)$ *are both concave functions.*

§ 3. A new characterization of the sphere and the isoperimetric problem

3.1. A characterization of the sphere

The characteristic function of the sphere is equal to 4π over the whole interval $0 \leqq t \leqq r$. However, there are other surfaces whose characteristic functions have the same property. For example, the convex cover of a torus, the spherical cylinder which is defined as a convex cover of two equal spheres whose centres are distinct, etc.

Further the Theorem 19 signifies that among the class of edge-singular convex polyhedra there exist some examples whose characteristic functions are constantly equal to 4π over their intervals. Here with the object of characterization of the sphere by using the characteristic function let us begin the following lemma.

Lemma. *Of all the closed convex surfaces having a constant characteristic function over the whole interval of its interior parallel sequence, the cap-surface and the tangential surface of a sphere are the one having a point-kernel.*

Proof. The condition that

$$\varkappa(t) = \text{const.} \qquad 0 \leqq t \leqq r.$$

requires that all the extreme supporting planes do not vanish over the whole interval, $0\leqq t\leqq r$. Moreover in order to have a point-kernel, the distance from the point-kernel $K(0)$ to each of the extreme supporting planes of $K(t)$ must be equal to t. Hence the necessary and sufficient condition for the property ist hat: *all the extreme supporting planes of K are tangent planes of a sphere with radius r.* Therefore the surface satisfying the given condition is the cap-surface of a sphere or the tangential surface of a sphere only. Thus the lemma has been proved.

Further we have

Corollary. *If $K(t)$ is an interior parallel surface of a closed convex surface K with inradius r (>0) and has a point-kernel and its characteristic function $\varkappa(t)$ is constantly equal to c in $0\leqq t\leqq r$, then we have*

$$V(t)=\frac{c}{3}t^3,\qquad S(t)=ct^2,\qquad M^*(t)=ct \qquad\qquad 0\leqq t\leqq r$$

where c is equal to the characteristic function of K.

Theorem 24. *Of all the closed convex surfaces whose characteristics are equal to 4π over the whole interval of its interior parallel sequence, the sphere is the one having a point-kernel.*

Proof. It is clear that the sphere satisfies the given condition. On the other hand, if K is a convex body with the given property, by the lemma it should be a cap-surface of a sphere or a tangential surface of a sphere.

First, let K be a cap-surface of a sphere in the strict sense. Then $K(t)$ should always contain one vertex at the very least and its form-figure $\Pi(t)$ should always contain one or more vertices which correspond to the ones of $K(t)$. Then since such a form-fiigure $\Pi(t)$ includes the unit-sphere within it in the strict sense, it follows that

$$\varkappa(t)>4\pi \qquad\qquad 0\leqq t\leqq r.$$

Hence, the cap-surface of a sphere in the strict sense does not satisfy the given condition.

Next, let K be a tangential-surface of a sphere. Then we may distinguish two cases according to whether K is edge-regular or edge-singular.

First, let K be an edge-singular tangential surface of a sphere of radius r (>0). Since the characteristic function $\varkappa(t)$ should be equal to 4π constantly over the interval $0\leqq t\leqq r$, none of its negative edges vanishes. Hence the negative edges converge to the line-segments of finite-length as $t\to 0$. Then the sequence of the interior parallel surface of the surface K cannot have a point-kernel. Therefore K is not edge-singular.

Next, If K is an edge-regular tangential surface of a sphere of radius $r(>0)$, let us denote one of its edges by B. Then K has two extreme supporting planes, say $\overline{A}_1$ and $\overline{A}_2$, which meet at the edge B. Hence the form-figure Π contains two image planes $\overline{\Omega}_1$ and $\overline{\Omega}_2$ which correspond to $\overline{A}_1$ and $\overline{A}_2$ respectively. Then the form-figure Π contains anyway one edge which is defined by $\overline{\Omega}_1$ and $\overline{\Omega}_2$. Therefore its characteristic $\varkappa$ is greater than 4π. After all it follows that K can not contain any edges. Hence the convex surface which is satisfied with the condition should be a sphere. Thus the theorem has been completely proved.

3.2. The Brunn-Minkowski theorerm

i) A set of isoperimetric inequalities

The isoperimetric problem in E_3 is usually stated as follows: *Of all convex bodies with a given volume, the sphere has the least surface area; of all convex bodies with a given surface area, the sphere has the greatest volume.* The isoperimetric property of the sphere is usually proved by the isoperimetric inequality

$$S^3 - 36\pi V^2 \geqq 0.$$

As is well-known, H. Minkowski resolved it into two inequalities

$$M^2 - 4\pi S \geqq 0, \qquad S^2 - 3MV \geqq 0.$$

Further, in connection with the solutions of these inequalities, various inequalities have been added. We treated a set of isoperimetric inequalities in the previous papers [9] and [10] in such a case where the characteristic function $\varkappa(t)$ is a monotone decreasing step-function. Here, let us prove some of the inequalities by applying the Brünn-Minkowsk theorem.

ii) A concave array (linear Eikörperschar)

Let K_0 and K_1 be two convex bodies in E_3 and let λ be such a parameter that $0 \leqq \lambda \leqq 1$. Then, the class of convex bodies which are defined by

$$K_\lambda = (1-\lambda)K_0 + \lambda K_1 \tag{40}$$

is said to form a *concave array*. Let the volumes of the bodies K_0, K_1, and K_λ be V_0, V_1 and V_λ respectively. Then, the Brunn-Minkowski theorem in E_3 is expressed as follows: *if $K_\lambda = (1-\lambda)K_0 + \lambda K_1$, then the third root of the volume V_λ of K_λ is a concave function with respect to λ. That is to say, we have*

$$\sqrt[3]{V_\lambda} \geqq (1-\lambda)\sqrt[3]{V_0} + \lambda\sqrt[3]{V_1}$$

where the equality holds if and only if K_0 and K_1 are homothetic.

Now let K_0 be the form-figure of K_1. Then it is easy to see that K_λ is the interior parallel body of K_1 at the distance $\lambda(\geqq 0)$ from K_1 and K_0 is the form-figure of K_λ too. Here let us consider that K_1 has the inradius $t(>0)$ and the two bodies K_λ and K_1 are represented by $K(t)$ and $K(t+\lambda)$ respectively. Then $K(t)$ is an interior parallel body of the convex body $K(t+\lambda)$ at the distance λ from $K(t+\lambda)$, and $K(t)$ and $K(t+\lambda)$ have the common form-figure K_0. Hence, we have the extended Steiner's formula for the volume of the interior parallel surface[4)]

$$V(t+\lambda) = V(t) + S(t)\,\lambda + M^*(t)\,\lambda^2 + \frac{1}{3}\varkappa(t)\lambda^3 \tag{41}$$

where $V(t)$, $S(t)$, $M^*(t)$ and $\varkappa(t)$ are the volume, surface area, total mean curvature and characteristic function of $K(t)$ respectively.

Next, if K_0 in (40) be a unit sphere, K_1 is the exterior parallel body of K_λ at the distance λ from K_λ. If we denote the volume of K_1 by $V\{t+\lambda\}$, by the well-known Steiner's formula, we have

$$V\{t+\lambda\} = V(t) + S(t)\,\lambda + M(t)\,\lambda^2 + \frac{4\pi}{3}\lambda^3 \tag{42}$$

where $V(t)$, $S(t)$ and $M(t)$ are the volume, surface area and integral of mean-curvature of the convex body K_λ in (40).

It is noteworthy that the Brunn-Minkowski theorem can be applied to the above formulas (41) and (42).

iii) Applications

First, taking up the third roots of both sides of the formuls (41) above obtained, we readily have their second derivatives with respect to λ as follows,

$$\frac{d^2}{d\lambda^2}(V(t+\lambda)^{\frac{1}{3}}) = -\frac{2}{9}[V(t+\lambda)]^{-\frac{5}{3}}[(S(t)^2-3M^*(t)V(t))+(M^*(t)S(t)-3\varkappa(t)V(t))\lambda$$
$$+(M^*(t)^2-\varkappa(t)S(t))\lambda^2].$$

Since the third root of the volume $V(t+\lambda)$ is a convex function of λ, it follows by the Brunn-Minkowski theorem that

$$\frac{d^2}{d\lambda^2}(V(t+\lambda)^{\frac{1}{3}}) \leqq 0.$$

Then, we have by using the right side of the above formula

$$M^*(t)^2 - \varkappa(t)S(t) \geqq 0 \tag{43}$$

and

$$S(t)^2 - 3M^*(t)V(t) \geqq 0. \tag{44}$$

The equalities in (43) *and* (44) *hold if and only if the convex body* $K(t)$ *and its form-figure* $\Pi(t)$ *are homothetic, that is to say,* $K(t)$ *is a cap-surface of a sphere or a*

4) It should be noted that even if $K(t)$ belongs to an interior parallel sequence $\{K(t): 0 \leqq t \leqq r\}$, the surface $K(t+\lambda)$ does not always belong to the sequence.

tangential surface of a sphere of radius t $(\geqq 0)$.

These results hold all over the interval $0 \leqq t \leqq r$ of the intrinsic interior parallel sequence of a convex body K with inradius r.

In the same manner, we can obtain the following inequalities from Steiner's formula (42) for the volume of the exterior parallel sequence.

$$M(t)^2 - 4\pi S(t) \geqq 0 \tag{45}$$

and

$$S(t)^2 - 3M(t)V(t) \geqq 0. \tag{46}$$

The equalities in (45) *and* (46) *hold if and only if the convex body* K_1 *is homothetic with the unit sphere* K_0. That is to say, *the equalities in* (45) *and* (46) *hold if and only if* K_1 *is a sphere of radius* t.

However, remembering the relation $M(t) \leqq M^*(t)$ and comparing (46) with (44), we can say that *the equality in* (46) *holds if and only if the convex body* $K_1(=K(t))$ *is a cap-surface of a sphere or a tangential surface of a sphere.*

iv) Schwarz's inequality

By eliminating $M(t)$ from formulas (45) and (46), we get the original isoperimetric inequality, that is to say, Schwarz's inequality:

$$S(t)^3 - 36\pi V(t)^2 \geqq 0. \tag{47}$$

The equality holds if and only if $K(t)$ *is a sphere and the inequality holds in* $0 \leqq t < \infty$.

In the same way from (43) and (44) we obtain in $0 \leqq t \leqq r$

$$S(t)^3 - 9\varkappa(t)V(t)^2 \geqq 0 \tag{48}$$

where the equality holds for the cap-surface of a sphere and the tangential surface of a sphere.

3.3. The explicit representation of the isoperimetric deficiencies and a set of the isoperimetric inequalities

i) In the same manner as in [9] we can give the explicit integral representations of the isoperimetric deficiencies in (43), (45), (44), (46), (47) and (48) as follows:

$$M^*(t)^2 - \varkappa(t)S(t) = 2\int_0^t M^*(s)(\varkappa(s)-\varkappa(t))du + S(0)(\varkappa(0)-\varkappa(t)) + M^*(0)^2 - \varkappa(0)S(0), \tag{49}$$

$$M(t)^2-4\pi S(t)=2\int_0^t(\mu(s)M(s)-4\pi M^*(s))ds+M(0)^2-4\pi S(0), \tag{50}$$

$$S(t)^2-3M^*(t)V(t)=\int_0^t(M^*(s)S(s)-3\varkappa(s)V(s))ds+S(0)^2, \tag{51}$$

$$S(t)^2-3M(t)V(t)=\int_0^t(M^*(s)S(s)-3\mu(s)V(s))ds+3\int_0^t S(s)(M^*(s)-M(s))ds \\ +S(0)^2, \tag{52}$$

$$S(t)^3-36\pi V(t)^2=6\int_0^t S(s)(M^*(s)S(s)-12\pi V(s))ds+18\int_0^t S(s)V(s) \\ \times(\varkappa(s)-4\pi)ds+S(0)^3, \tag{53}$$

$$S(t)^3-9\varkappa(t)V(t)^2=6\int_0^t S(s)(M^*(s)S(s)-3\varkappa(s)V(s))ds+18\int_0^t S(s)V(s) \\ \times(\varkappa(s)-\varkappa(t))ds+S(0)^3. \tag{54}$$

ii) Next let us consider a case where *the characteristic function $\varkappa(t)$ is a monotone decreasing function in the whole interval $0\leqq t\leqq r$.* Then we can obtain such a set of the new isoperimetric inequalities with the integration.

For example, taking the expression (49) and the formula (38), we have

$$M^*(t)^2-\varkappa(t)S(t)\geqq 2\int_0^t(\varkappa(u)-\varkappa(t))\left\{\int_0^u\varkappa(v)dv\right\}du+M^*(0)^2-\varkappa(0)S(0),$$

or

$$M^*(t)^2-\varkappa(t)S(t)-\left\{M^*(0)^2-\varkappa(0)S(0)\right\}\geqq 2\int_0^t(\varkappa(u)-\varkappa(t))\left\{\int_0^u(\varkappa(v)-\varkappa(t))dv\right\}du.$$

Here, since the expression at the right side is equal to the expression $\left\{\int_0^t(\varkappa(s)-\varkappa(t))ds\right\}^2$, we have

$$M^*(t)^2-\varkappa(t)S(t)-\left\{M^*(0)^2-\varkappa(0)S(0)\right\}\geqq\left\{\int_0^t(\varkappa(s)-\varkappa(t))ds\right\}^2,$$

or

$$M^*(t)^2-\varkappa(t)S(t)\geqq\left\{\int_0^t(\varkappa(s)-\varkappa(t))ds\right\}^2, \tag{55}$$

where $0\leqq t\leqq r$.

Here we can readily see that *the equality in* (55) *holds for the cap-surface of a sphere and the tangential surface of a sphere only.*

In the same way, employing the explicit representation of the isoperimetric deficiencies which have been obtained in the paper [9], we can express a set of the isoperimetric inequalities by means of the integration respectively. Let us put them together in the following theorem.

Theorem 25. *If the characteristic function $\varkappa(t)$ of the interior parallel sequence $\{K(t);\ 0\leqq t\leqq r\}$ is a monotone decreasing function in $0\leqq t\leqq r$, it follows that*

(55) $$M^*(t)^2-\varkappa(t)S(t)\geqq\left\{\int_0^t(\varkappa(s)-\varkappa(t))ds\right\}^2,$$

(56) $$M^*(t)^2-\mu(t)S(t)\geqq\left\{\int_0^t(\varkappa(s)-\mu(t))ds\right\}^2,$$

(57) $$M^*(t)^2-4\pi S(t)\geqq\left\{\int_0^t(\varkappa(s)-4\pi)ds\right\}^2,$$

(58) $$M^*(t)S(t)-3\varkappa(t)V(t)\geqq 4\int_0^t\int_0^s M^*(u)(\varkappa(u)-\varkappa(t))duds,$$

(59) $$M^*(t)S(t)-3\mu(t)V(t)\geqq 4\int_0^t\int_0^s M^*(u)(\varkappa(u)-\mu(t))duds,$$

(60) $$M^*(t)S(t)-12\pi V(t)\geqq 4\int_0^t\int_0^s M^*(u)(\varkappa(u)-4\pi)duds,$$

(61) $$S(t)^3-9\varkappa(t)V(t)^2\geqq 24\int_0^t S(s)\int_0^s\int_0^u M^*(v)(\varkappa(v)-\varkappa(s))dvduds,$$

(62) $$S(t)^3-9\mu(t)V(t)^2\geqq 24\int_0^t S(s)\int_0^s\int_0^u M^*(v)(\varkappa(v)-\mu(s))dvduds,$$

(63) $$S(t)^3-36\pi V(t)^2\geqq 24\int_0^t S(s)\int_0^s\int_0^u M^*(v)(\varkappa(v)-4\pi)dvduds,$$

(64) $$S(t)^3-3M^*(t)V(t)\geqq 4\int_0^t\int_0^s\int_0^u M^*(v)(\varkappa(v)-\varkappa(s))dvduds,$$

(65) $$S(t)^3-3M(t)V(t)\geqq 4\int_0^t\int_0^s\int_0^u M^*(v)(\varkappa(v)-\varkappa(s))dvduds.$$

Here, the equalities in (55), (58), (61) and (64) hold for the cap-surface of a sphere and the tangential surface of a sphere only. The equalities in (56), (59), (62) *and* (65) *hold for the cap-surface of a sphere only and those in* (75), (60) *and* (63) *hold for the sphere only.*

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ERRATA

Parallel series to a closed convex curve and surface and the differentiability of their quantities
by Shigeru Ohshio
The Science Reports of the Kanazawa University, 6 (1958), 15-24.

In both cases of the interior parallel sequences $\{C(t) : -r \leqq t \leqq 0\}$ of a closed convex curve C, and $\{K(t) : -r \leqq t \leqq 0\}$ of a closed convex surface K, their critical values of the parameter t constitute a countable set respectively but can not be generally enumerated in the order of the magnitude.

Corrections to the paper "A New Characterization of the Sphere and the Isoperimetric Problem in E_3"

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Shigeru Ohshio

The following corrections should be made :
P. 26, line 11. Insert "not generally" between the words "are" and "integrable."
P. 26, "Theorem 21" and PP. 31–33, "the integral representations in § 3.3" should be assumed that $V(t)$, $S(t)$, $M^*(t)$ and $M(t)$ are absolutely continuous in the interval $[0, t]$ and the formulas (30), (31) and (32) hold in the interval.