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Notes on Gap Theorems

by

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1. Introduction

Let $f(x)$ be an L^2-integrable and periodic function with period 1, and satisfy the following conditions,

(1. 1) $$\int_0^1 f(x)dx=0, \qquad \int_0^1 f(x)^2 dx=1,$$

and moreover let $R(n)$ denote

(1. 2) $$R(n)=\left(\int_0^1 |f(x)-s_n(x)|^2 dx\right)^{1/2},$$

where $s_n(x)$ is the n-th partial sum of the Fourier series of $f(x)$.

In my previous paper [1] we proved the following two theorems.

Theorem A. If for any $\alpha>0$ and for any integral number $p\geqq 1$,

(1. 3) $$R(n)=O(1/(\log_p n)^\alpha),$$

then the series

$$\sum \frac{f(n_k x)}{k \log k \log_2 k \cdots\cdots (\log_p k)^{1+\alpha}(\log_{p+1} k)^\beta},$$

converges at almost every points in the interval (0, 1), where the sequence $\{n_k\}$ satisfies that

(1. 4) $$\begin{cases} \text{(i)} & n_k \uparrow \infty, \\ \text{(ii)} & \text{it is the sequence of integral numbers.} \end{cases}$$

Theorem B. If for any $0<\alpha\leqq\frac{1}{2}$,

(1. 5) $$R(n)=O(1/n^\alpha),$$

then the convergence of

$$\sum |a_k|^2\sqrt{k}(\log k)^2<\infty,$$

implies the almost everywhere convergence of a gap series

(1. 6) $$\sum a_k f(n_k x),$$

where its gap conditions are given by (1. 4).

In this note we discuss the almost everywhere convergence of the series (1. 6) satisfying the gap conditions (1. 4), and we prove two theorems, each of which is the extension of Theorem A and Theorem B respectively.

Theorem 1. If for any $\alpha>0$ and for any integral number $p\geqq 1$, $R(n)$ satisfies (1. 3), then for the almost everywhere convergence of (1. 6) satisfying the gap conditions

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(1. 4), it is sufficient that

(1. 7) $$\sum |a_k|^2\ k \log k \log_2 k \cdots\cdots (\log_p k)^{1-2\alpha}(\log_{p+1} k)^2 < \infty.$$

Theorem 2. If for any $\alpha>0$, $R(n)$ satisfies (1. 5), then for the almost everywhere convergence of (1. 6) satisfying the gap conditions (1. 4), it is sufficient that

(a) if $0<\alpha\leqq\frac{1}{2}$, then

(1. 8) $$\sum |a_k|^2\ k^{1/\mu}\ (\log k)^2 < \infty,$$

(b)*) if $\frac{1}{2}\leqq\alpha$, then

(1. 9) $$\sum |a_k|^2\ k^{1/2}\ (\log k)^2 < \infty,$$

where

(1. 10) $$\mu = 1+2\alpha.$$

For the proof of these theorems, we make use of the several results obtained in [1]. If for integral numbers Z and N, $Z+1\leqq n_j < n_k \leqq Z+N$, then it follows that

(1. 11) $$\left|\int_0^1 f(n_j x) f(n_k x)\, dx\right| \leqq R\left(\frac{Z}{N}\right)^2,$$

from which

(1. 12) $$\int_0^1 \left|\sum_{k=Z+1}^{Z+N} a_k f(n_k x)\right|^2 dx \leqq \left(1+NR^2\left(\frac{Z}{N}\right)\right)\left(\sum_{k=Z+1}^{Z+N} |a_k|^2\right).$$

2. Proof of Theorem 1

If we put in (1. 12)

$$Z=s^2 \text{ and } N=(s+1)^2-s^2=(2s+1)(\equiv N(s)),$$

then

$$1+NR^2\left(\frac{Z}{N}\right)=1+O(s)O\left(\frac{1}{(\log_p s)^{2\alpha}}\right)=O\left(s(\log_p s)^{-2\alpha}\right).$$

Hence, from the convergence of the series

(2. 1) $$\sum_{s=1}^{\infty}\left(\sum_{k=s^2+1}^{(s+1)^2}|a_k|^2\right)^{1/2}\frac{\sqrt{s}}{(\log_p s)^{\alpha}},$$

it follows that the almost everywhere convergence of

(2. 2) $$\lim_{s\to\infty}\sum_{k=1}^{s^2} a_k f(n_k x).$$

To verify the truth of (2. 1), we put

$$g(k)=k\log k\log_2 k \cdots\cdots (\log_p k)^{1-2\alpha}(\log_{p+1} k)^2,$$

then

$$\sum_{s=1}^{\infty}\frac{\sqrt{s}}{(\log_p s)^{\alpha}}\left(\sum_{k=s^2+1}^{(s+1)^2}|a_k|^2\right)^{1/2}=\sum_{s=1}^{\infty}\frac{\sqrt{s}}{(\log_p s)^{\alpha}}\left(\sum_{k=s^2+1}^{(s+1)^2}|a_k|^2\ g(k)\frac{1}{g(k)}\right)^{1/2}$$

$$\leqq\sum_{s=1}^{\infty}\frac{\sqrt{s}}{(\log_p s)^{\alpha}}\ \frac{1}{\sqrt{g(s^2)}}\ \left(\sum_{k=s^2+1}^{(s+1)^2}|a_k|^2\ \ g(k)\right)^{1/2}$$

*) This case was already proved in [1], so we consider the case (a) only.

$$\leqq \left(\sum_{s=1}^{\infty} \frac{s}{(\log_p s)^{2\alpha}} \frac{1}{g(s^2)} \right)^{1/2} \left(\sum_{s=1}^{\infty} \sum_{k=s^2+1}^{(s+1)^2} |a_k|^2 g(k) \right)^{1/2}.$$

From (1. 7) and the above last inequality, we obtain (2. 1) and from it the almost everywhere convergence of (2. 2) holds.

Now for our purpose, we try to prove the convergence of series

$$\text{(2. 3)} \qquad \sum_{s=1}^{\infty} \int_0^1 \max_{s^2 < m < (s+1)^2} \left| \sum_{k=s^2+1}^{m} a_k f(n_k x) \right|^2 dx < \infty.$$

However the general term of the series (2. 3) is less than

$$O(1) \log N(s) \sum_{v=1}^{[\log N(s)]} \sum_{u=0}^{[N(s)2^{-v}]} \int_0^1 \left| \sum_{k=s^2+u2^v+1}^{s^2+(u+1)2^v} a_k f(n_k x) \right|^2 dx$$

$$= O(1) \log N(s) \sum_{v=1}^{[\log N(s)]} 2^{-v} \ N(s) \sum_{k=s^2+1}^{(s+1)^2} |a_k|^2 \left(1 + 2^v \left(\log_p \frac{s^2}{2^v} \right)^{-2\alpha} \right)$$

$$\text{(2. 4)} \qquad \leqq O(1) \left(\sum_{k=s^2+1}^{(s+1)^2} |a_k|^2 \right) N(s) \log N(s) \sum_{s=1}^{[\log N(s)]} \left(\log_p \frac{s^2}{2^v} \right)^{-2\alpha}.$$

On the other hand the last term is less than

$$O(1) \sum_{v=1}^{[\log N(s)]} \left(\log_p s^2 N(s)^{-1} \right)^{-2\alpha} = O(1) \sum_{v=1}^{[\log N(s)]} (\log_p s)^{-2\alpha}$$

$$\text{(2. 5)} \qquad \leqq O(1) \ (\log s) \ (\log_p s)^{-2\alpha}.$$

From (2. 4) and (2. 5) it holds that the general term of series (2. 3) is less than

$$O(1) \sum_{k=s^2+1}^{(s+1)^2} |a_k|^2 s \log s \frac{\log s}{(\log_p s)^{2\alpha}}$$

$$\text{(2. 6)} \qquad \leqq \sum_{k=s^2+1}^{(s+1)^2} |a_k|^2 g(k) \frac{\log k}{\sqrt{k} \ (\log_{p+1} k)^2} \leqq O(1) \sum_{k=s^2+1}^{(s+1)^2} |a_k|^2 g(k)$$

This and the hypothesis (1. 7) of Theorem 1 deduce the proof of Theorem 1.

3. Proof of Theorem 2

If we put in (1. 12) by use of μ defined by (1. 10),

$$Z = s^{\mu}, \text{ and } N = (s+1)^{\mu} - s^{\mu} (\equiv N(s)),$$

we can immediately verify that the convergence of

$$\text{(3. 1)} \qquad \sum_{s=1}^{\infty} \left(\sum_{k=s^{\mu}+1}^{(s+1)^{\mu}} |a_k|^2 \right)^{1/2} < \infty$$

implies the almost everywhere convergence of

$$\text{(3. 2)} \qquad \lim_{s \to \infty} \sum_{k=1}^{s^{\mu}} a_k f(n_k x).$$

However (3. 1) is easily verified, i. e.

$$\sum_{s=1}^{\infty} \left(\sum_{k=s^{\mu}+1}^{(s+1)^{\mu}} |a_k|^2 \right)^{1/2} = \sum_{s=1}^{\infty} \left(\sum_{k=s^{\mu}+1}^{(s+1)^{\mu}} |a_k|^2 \ k^{1/\mu} \ (\log k)^2 \frac{1}{k^{1/\mu} (\log k)^2} \right)^{1/2}$$

$$\leqq \sum_{s=1}^{\infty} \frac{1}{\sqrt{s}(\log s)^2}\left(\sum_{k=s^{\mu}+1}^{(s+1)^{\mu}} |a_k|^2\ k^{1/\mu}\ (\log k)^2\right)^{1/2}$$

$$\leqq \left(\sum_{s=1}^{\infty} \frac{1}{s(\log s)^2}\right)^{1/2} \left(\sum_{s=1}^{\infty}\sum_{k=s^{\mu}+1}^{(s+1)^{\mu}} |a_k|^2\ k^{1/\mu}\ (\log k)^2\right)^{1/2}$$

$$\leqq O(1)\ \left(\sum_{k=1}^{\infty} |a_k|^2\ k^{1/\mu}\ (\log k)^2\right)^{1/2}.$$

In the case (a), evidently the above series converges, and then we obtain the convergence of (3. 2).

Lastly for the completion of the proof of Theorem 2, it is sufficient to prove that

(3. 3) $$\sum_{s=1}^{\infty}\int_0^1 \max_{s^{\mu}<m<(s+1)^{\mu}} \left|\sum_{k=s^{\mu}+1}^{m} a_k\, f(n_k\, x)\right|^2 dx<\infty.$$

The general term of (3. 3) is less than

$$O(1)\ \log\ N(s) \sum_{v=1}^{[\log N(s)]} \sum_{u=0}^{[N(s)2^{-v}]} \int_0^1 \left|\sum_{k=s^{\mu}+u2^v+1}^{s^{\mu}+(u+1)2^v} a_k\, f(n_k\ x)\right|^2 dx$$

$$\leqq O(1)\ \log\ N(s) \sum_{v=1}^{[\log N(s)]} \sum_{u=0}^{[N(s)2^{-v}]} \sum_{k=s^{\mu}+1}^{(s+1)^{\mu}} |a_k|^2 (1+2^{v(2\alpha+1)}s^{-2\alpha\mu})$$

(3. 4) $$\leqq\left(\sum_{k=s^{\mu}+1}^{(s+1)^{\mu}} |a_k|^2\right)\left(\log\ N(s)\right)\ N(s) \sum_{v=1}^{[\log N(s)]} 2^{-v}2^{(2\alpha+1)v}s^{-2\alpha\mu}$$

Since in the last term, it holds that

$$2^{v(2\alpha+1)}2^{-2\alpha\mu}=N(s)^{(2\alpha+1)}s^{-2\alpha\mu}=s^{(2\alpha+1)(\mu-1)-2\alpha\mu}=O(1),$$

(3. 4) is less than

(3 .5) $$O(1) \sum_{k=s^{\mu}+1}^{(s+1)^{\mu}} |a_k|^2\, N(s) \log N(s) \sum_{v=1}^{\infty} 2^{-v} \leqq \sum_{k=s^{\mu}+1}^{(s+1)^{\mu}} |a_k|^2 k^{1-1/\mu}\ \log\ k$$

Now if α satisfies the case (a) then

(3. 6) $$0<\alpha\leqq\frac{1}{2}\quad \text{and}\quad 1-\frac{1}{\mu}\leqq\frac{1}{\mu}.$$

(3. 5) and (3. 6) indicate the convergence of (3. 3), and we complete the proof of Theorem 1.

Reference

[1] N. Matsuyama, On the convergence of Some Gap Series, The Science Reports of Kanazawa University Vol. VI, Nomber 2, 1958.