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On the Gap Sequence Having Random Signs

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1. Introduction

In the present note let $f(x)$, $-\infty<x<+\infty$, will denote a Borel-measurable function satisfying the following conditions:

$$f(x+1)=f(x), \tag{1.1}$$

$$\int_0^1 f(x)dx=0, \tag{1.2}$$

and

$$\int_0^1 f^2(x)dx=1. \tag{1.3}$$

Further let $\{n_k\}$ be a lacunary sequence of positive integers.

The problem whether, for a gap sequence $\{f(n_k x)\}$, probability limit theorems hold or not is very difficult if n_k does not divide n_{k+1}.

In this note we consider a gap sequence

$$\{f(n_k x)\ \varphi_k(t)\},$$

where $\{\varphi_k(t)\}$ is the system of Rademacher functions and prove the following theorems.

Theorem 1. If $f(x)$ satisfies, for $\alpha>1$ and $h\to 0$,

$$\left(\int_0^1 |f(x)-f(x+h)|^4 dx\right)^{1/4} = O(1/|log_2|h||^\alpha) \tag{1.4}$$

then we have, for almost all t,

$$\overline{lim}_{N\to\infty} \frac{\sum_1^N f(n_k x)\ \varphi_k(t)}{\sqrt{2N\ log_2 N}}=1, \qquad \text{p.p. in } x.$$

Theorem 2. If $f(x)$ satisfies

$$|f(x)|\leqq M \tag{1.5}$$

and, for $\alpha>0$ and $h\to 0$,

$$\int_0^1 |f(x)-f(x+h)|\,dx = O(1/|\,log\,|h||^{\alpha}), \tag{1.6}$$

then we have, for almost all t,

$$\lim_{N\to\infty}\left|\left(x;\ \frac{1}{\sqrt{N}}\sum_{k=1}^N f(n_k x)\ \varphi_k(t)\leqq y\right)\right| = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^y e^{-\frac{u^2}{2}}du.$$

We can prove the above theorems in the same way as that of [1].

2. Some Lemmas

Lemma 1. Under the hypotheses of Theorem 1, we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N} f^2(n_k x)=1, \qquad \text{p.p. in } x.$$

Proof. By (1.4), we have

$$\left\{\int_0^1 |f^2(x)-f^2(x+h)|^2\,dx\right\}^{1/2}=O\left(\int_0^1 |f(x)-f(x+h)|^4\,dx\right)^{1/4}$$
$$=O(1/|\log|h||^{\alpha}),$$

as $h\to 0$. Hence by (1.3) and the theorem of P. Erdös [2], we can prove Lemma 1.

Lemma 2. Under the hypotheses of Theorem 2, we have

$$\int_0^1 f(n_k x) f(n_j x)dx=O\left(\frac{1}{|k-j|^{\alpha}}\right), \qquad (|k-j|\to+\infty).$$

This lemma is a special case of that of S. Izumi [3].

Lemma 3. (c.f. [4]) Let z_k, $1\leqq k\leqq N$, be a sequence of independent random variables and $E(z_k)=0$, $E(z_k^2)<+\infty$. If μ and x satisfy

$$\mathop{Max}_{1\leqq k\leqq N}|z_k|\leqq\mu,$$

and

$$\mu x\geqq\sum_1^N E(z_k^2),$$

then we have

$$Prob\left(\left|\sum_{k=1}^N z_k\right|>x\right)\leqq 2e^{-x/4\mu}$$

3. Proof of Theorem 1

In this section we prove Theorem 1.

Let us put, for $k=1, 2, \cdots\cdots$,

$$(3.1) \qquad f_k(x)=\begin{cases} f(x), & if\ |f(x)|\leqq k^{1/4},\\ 0, & if\ otherwise,\end{cases}$$

and

$$(3.2) \qquad f_k'(x)=f(x)-f_k(x).$$

Then we have

$$\sum_{k=1}^{\infty}|(x;\ f_k'(n_k x)\neq 0)|=\sum_{k=1}^{\infty}|(x;\ |f(x)|\geqq k^{1/4})|=O\left(\int_0^1|f(x)|^4dx\right)=O(1).$$

From this, we have

$$(3.3) \qquad \sum_{k=1}^{\infty}|f_k'(n_k x)|<+\infty, \qquad \text{p.p. in } x.$$

By Lemma 1 and the above relations, we have

$$(3.4) \qquad \sum_{k=1}^N f_k^2(n_k x)=\sum_{k=1}^N f^2(n_k x)-2\sum_{k=1}^N f_k(n_k x)f_k'(n_k x)+\sum_{k=1}^N f_k'^2(n_k x)$$
$$=N+o(N)+O(N^{1/4})+O(1)=N+o(N),$$

p.p. in x as $N\to\infty$.

Now for each x, $\{f_k(n_k x)\ \varphi_k(t)\}$ is a sequence of independent random variables and

(3.5) $$E\{f_k(n_k x)\varphi_k(t)\}=0,$$

and

(3.6) $$\sum_1^N E\left\{(f_k(n_k x)\varphi_k(t))^2\right\}=\sum_1^N f_k^2(n_k x).$$

Furthermore from (3.4) and (3.6) we have, for almost all x,

$$|f_N(n_N x)\varphi_N(t)|\leqq N^{1/4}=o\left\{\left(\frac{\sum_1^N f_k^2(n_k x)}{log_2\left(\sum_1^N f_k^2(n_k x)\right)}\right)^{1/2}\right\}$$

as $N\to\infty$. Hence, by the so-called law of the iterated logarithm, we have, for almost all x,

$$\overline{lim}_{N\to\infty}\frac{\sum_1^N f_k(n_k x)\varphi_k(t)}{\sqrt{2N\,log_2 N}}=1, \qquad \text{p.p. in } x.$$

Since $f_k(n_k x)$ is Borel-measurable, we have, for almost all t,

(3.7) $$\overline{lim}_{N\to\infty}\frac{\sum_1^N f_k(n_k x)\varphi_k(t)}{\sqrt{2N\,log_2 N}}=1, \qquad \text{p.p. in } x.$$

By (3.3) and (3.7), we can prove Theorem 1.

4. Proof of Theorem 2

In this section we prove Theorem 2.

Let us put, for $N=1, 2, \cdots\cdots$,

$$S_N=S_N(x,t)=\sum_1^N f(n_k x)\varphi_k(t).$$

For the proof of Theorem 2, it is sufficient to show that

(4.1) $$\psi_N(\lambda,t)=\int_0^1 exp\left(\frac{i\lambda S_N}{\sqrt{N}}\right)dx\to e^{-\lambda^2/2},$$

p.p. in t as $N\to+\infty$, holds uniformly in λ which is contained in any finite interval. In the following, we may assume that

(4.2) $$|\lambda|\leqq A.$$

Using the relation

$$e^x=(1+x)e^{-\frac{x^2}{2}+O(|x|^3)}$$

as $x\to 0$, we have

$$\psi_N(\lambda,t)=\int_0^1\prod_1^N\left(1+\frac{i\lambda f(n_k x)}{\sqrt{N}}\varphi_k(t)\right)e^{-\xi_N(x)+\eta_N(x)}\,dx,$$

where

$$\xi_N(x)=\frac{\lambda^2}{2N}\sum_1^N f^2(n_k x),$$

and

$$\eta_N(x)=\frac{|\lambda|^3}{N^{3/2}}\sum_1^N|f(n_k x)|^3.$$

Then we have, by (1.5) and Lemma 1,

$$\lim_{N\to\infty}\xi_N(x)=\frac{\lambda^2}{2} \qquad \text{p.p. in } x,$$

and

$$\lim_{N\to\infty}\eta_N(x)=0 \qquad \text{for all } x,$$

and the convergence is uniform in λ. Further we have

$$\left|\prod_1^N\left(1+\frac{i\lambda f(n_kx)\varphi_k(t)}{\sqrt{N}}\right)^2\right|$$

$$\leqq\prod_1^N\left(1+\frac{\lambda^2M^2}{N}\right)\leqq e^{A^2M^2},$$

and

$$|\xi_N(x)|\leqq A^2M^2.$$

Hence we have

$$\left|\int_0^1\prod_1^N\left(1+\frac{i\lambda f(n_kx)}{\sqrt{N}}\varphi_k(t)\right)e^{-\lambda^2/2}dx-\psi_N(\lambda,t)\right|$$

$$\leqq\left\{\int_0^1\left|\prod_1^N\left(1+\frac{i\lambda f(n_kx)}{\sqrt{N}}\varphi_k(t)\right)\right|^2dx\right\}^{1/2}\left\{\int_0^1\left|e^{-\lambda^2/2}-e^{-\xi_N(x)+\eta_N(x)}\right|^2dx\right\}^{1/2}=o(1),$$

uniformly in λ and t as $N\to+\infty$. Hence for (4.1), it is sufficient to prove that

$$\lim_{N\to\infty}\int_0^1\Pi_N(x,t)dx=1 \qquad \text{p.p. in } t, \tag{4.3}$$

holds uniformly in λ, where

$$\Pi_N(x,t)=\prod_1^N\left(1+\frac{i\lambda f(n_kx)\varphi_k(t)}{\sqrt{N}}\right).$$

If we put

$$J_N(t)=\int_0^1\Pi_N(x,t)dx-1,$$

then we have

$$\int_0^1|J_N(t)|^2dt=\int_0^1\int_0^1dx\,dy\int_0^1(\Pi_N(x,t)-1)(\overline{\Pi}_N(y,t)-1)dt$$

$$=\int_0^1\int_0^1dx\,dy\int_0^1\Pi_N(x,t)\overline{\Pi}_N(y,t)dt-1$$

$$=\int_0^1\int_0^1dx\,dy\int_0^1\prod_1^N\left\{1+\frac{i\lambda\varphi_k(t)(f(n_kx)-f(n_ky))}{\sqrt{N}}+\frac{\lambda^2f(n_kx)f(n_ky)}{N}\right\}dt-1$$

$$=\int_0^1\int_0^1\left(1+\frac{\lambda^2f(n_kx)f(n_ky)}{N}\right)dx\,dy-1$$

$$\leqq\int_0^1\int_0^1\exp\left\{\frac{\lambda^2}{N}\sum_1^Nf(n_kx)f(n_ky)\right\}dx\,dy-1$$

$$=\int_0^1\int_0^1\Big[\frac{\lambda^2}{N}\sum_1^N f(n_kx)f(n_ky)+\frac{1}{2}\Big(\frac{\lambda^2}{N}\sum_1^N f(n_kx)f(n_ky)\Big)^2 exp\Big\{\frac{\eta\lambda^2}{N}\sum_1^N f(n_kx)f(n_ky)\Big\}\Big]dxdy,$$

where $0<\eta<1$.

By (1.2), (1.5) and (4.2), we have

$$\int_0^1|J_N(t)|^2dt\leqq\frac{A^2}{N^2}e^{A^2M^2}\int_0^1\int_0^1(\sum_1^N f(n_kx)f(n_ky))^2\,dx\,dy.$$

On the other hand we have, by Lemma 2

$$\frac{1}{N^2}\int_0^1\int_0^1(\sum_1^N f(n_kx)f(n_ky))^2\,dx\,dy$$

$$=\frac{1}{N}+\frac{2}{N^2}\sum_{k=1}^{N-1}\sum_{j=k+1}^{N}(\int_0^1 f(n_kx)f(n_jx)dx)^2=N^{-1}+O(N^{-2\alpha})=O(N^{-\beta}),$$

where

$$\beta=Min\,(1,2\alpha).$$

If we put

$$N_k=[k^{2/\beta}],$$

then

$$\sum_{k=1}^{\infty}\int_0^1|J_{N_k}(t)|^2dt<+\infty,$$

uniformly in λ. Hence (4.3) holds for $N=N_k$, $k=1, 2,\cdots\cdots$.

Next we put, for any fixed k ($k\geqq k_0$) and $1\leqq i<N_{k+1}-N_k$,

$$z_i=f(n_{N_k+i}x)\,\varphi\,N_{k+i}(t).$$

Then we have

$$E(z_i)=0,$$

$$\sum_{i=1}^{N_{k+1}-N_k-1}E(z_i^2)=\sum_{i=N_k+1}^{N_{k+1}-1}f^2(n_kx)$$

$$\leqq M^2(N_{k+1}-N_k)=O(k^{\frac{2}{\beta}-1}),$$

and, since $\beta\leqq1$,

$$\underset{1\leqq i<N_{k+1}-N_k}{Max}|z_i|\leqq M\leqq k^{\frac{1}{\beta}-\frac{1}{2}},$$

for $k\geqq k_0$. Hence we have, by Lemma 3,

$$Prob.(|\sum_{i=1}^N z_i|>k^{\frac{1}{\beta}-\frac{1}{3}})\leqq 2e^{-\frac{k^{1/6}}{4}},$$

for any N, $1\leqq N<N_{k+1}-N_k$.

From the above inequality, we have

$$\sum_{k\geqq k_0}Prob.(\max_{1\leqq N<N_{k+1}-N_k}|\sum_1^N z_i|>k^{\frac{1}{\beta}-\frac{1}{3}})$$

$$=\sum_{k=1}^{\infty}O(k^{\frac{2}{\beta}-1}e^{-\frac{k^{1/6}}{4}})=O(1).$$

This shows that, for all x,

$$\overline{\lim_{k\to\infty}} \operatorname*{Max}_{1\leq N<N_{k+1}-N_k} \left| \frac{\sum_{i=N_k+1}^{N_k+N} f(n_i x)\varphi_i(t)}{k^{\frac{1}{\beta}-\frac{1}{3}}} \right| \leq 1 \qquad \text{p.p. in } t.$$

Since $f(n_k x)$ is Borel-measurable and

$$k^{\frac{1}{\beta}-\frac{1}{3}} = o(\sqrt{N_k})$$

as $k\to\infty$, we have, for almost all t,

$$\overline{\lim_{k\to\infty}} \operatorname*{Max}_{1\leq N<N_{k+1}-N_k} \left| \frac{\sum_{i=N_k+1}^{N_k+N} f(n_i x)\varphi_i(t)}{\sqrt{N_k+N}} \right| = 0 \qquad \text{p.p. in } x. \tag{4.5}$$

By (4.5), we can prove Theorem 2.

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