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The Law of the Iterated Logarithm for Dependent Random Variables

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§ 1. Let

(1.1) $$\xi_1, \xi_2, \ldots$$

be a sequence of random variables defined on a probability space (Ω, B, P). We say that (1.1) is m-dependent if for some integer $m \geqq 0$, the inequalty $s-r>m$ implies that the two sets

$$(\xi_1, \xi_2, \ldots, \xi_{r-1}, \xi_r) \ (\xi_s, \xi_{s+1}, \ldots, \xi_{n-1}, \xi_n)$$

are independent.

By the induction, if (1.1) is m-dependent, then the following blocks

$$(\xi_{r_i}, \xi_{r_i+1}, \ldots, \xi_{r_i'-1}, \xi_{r_i'}), \ (i=1, 2, \ldots)$$

are independent whenever $r_{i+1}-r_i'>m$ and $r_i' \geqq r_i$ $(i=1, 2, \ldots)$. The notion of m-dependence is the most simple one that can be considered as the weakened independence.

For m-dependent random variables many authors proved the central limit theorem (c. f. [1], [2] and [3]) as an extension of the limit theorems of independent random variables. In this note we shall confine ourselves to the case of m-dependent sequence and prove the law of the iterated logarithm.

We suppose that, for all i

(1.2) $$E(\xi_i)=0,$$

and for an $\varepsilon>0$

(1.3) $$\int_{|\xi_i| \leqq n} \xi_i^2 \, dP = 0\left(\frac{1}{(\log n)^{1+\varepsilon}}\right) \qquad (n \to +\infty)$$

uniformly in i.

From (1.3) it follows that for each i there exists $E(\xi_i^2)$, and

(1.4) $$E(\xi_i^2)=0\,(1) \qquad (i \to +\infty).$$

Further let us put

(1.5) $$S_n=\sum_{i=1}^{n} \xi_i$$

and

(1.6) $$B_n=E(S_n^2).$$

In §§ 2—4, we prove the following

Theorem. If (1.2), (1.3) and

(1.7) $$\lim_{n \to \infty} \frac{1}{n} B_n = \sigma^2 > 0,$$

then we have

$$P\left(\overline{\lim_{n\to\infty}} \frac{S_n}{\sqrt{2B_n \log_2 B_n}}=1\right)=1.$$

For the proof of this theorem we use the well known theorem of A. N. Kolmogorov, which states the law of the iterated logarithm for independent random variables.

§ 2. For any fixed α $(1<\alpha<2)$, we put

(2.1) $$n_k=[k^\alpha],$$

and

(2.1′) $$m_k=n_{k+1}-(m+1)$$

for any integer $k\geqq k_0$, where $k_0^{\alpha-1}>m$.

Then we have

(2.2) $$n_{k+1}-n_k=m_k-m_{k-1}=O(k^{\alpha-1}) \qquad (k\to+\infty).$$

We define random variables as follows ;

(2.3) $$V_k\equiv\sum_{i=m_k+1}^{n_{k+1}-1}\xi_i$$

and

(2.3′) $$V_k'\equiv\underset{m_k<j<n_{k+1}}{Max}\left|\sum_{i=m_k+1}^{j}\xi_i\right|.$$

Then $\{V_k\}$ $(k\geqq k_0)$ is a sequence of independent random variables having the mean value zero and

$$V_k'\leqq V_k''\equiv\sum_{i=m_k+1}^{n_{k+1}-1}|\xi_i|.$$

From (1.4) we have, by the Minkowski's inequality

$$E(V_k^2)\leqq E(V_k'^2)\leqq E(V_k''^2)\leqq\left(\sum_{i=m_k+1}^{n_{k+1}-1}\{E(\xi_i^2)\}^{1/2}\right)=O(1) \qquad (k\to+\infty).$$

Therefore both series,

$$\sum_k\frac{E(V_k^2)}{k^\alpha \log_2 k} \text{ and } \sum_k E\left(\left(\frac{V'_k}{\sqrt{k^\alpha \log_2 k}}\right)^2\right)$$

are convegent. From the first, we have by the Khintchine-Kolmogorov's theorem

$$P\left(\sum_k\frac{V_k}{\sqrt{k^\alpha \log_2 k}} \text{ converges}\right)=1,$$

and this implies

$$P\left(\frac{1}{\sqrt{k^\alpha \log_2 k}}\sum_{i=k_0}^{k}V_i\to 0\right)=1.$$

And from the second we have

$$P\left(\sum_k\frac{(V_k')^2}{k^\alpha \log_2 k} \text{ converges}\right)=1,$$

and this implies

$$P\left(\frac{V'_k}{\sqrt{k^\alpha \log_2 k}}\to 0\right)=1.$$

Hence we have

(2.4) $$P\left(\frac{1}{\sqrt{k^\alpha \log_2 k}}\left(\sum_{i=k_0}^{k-1}V_i+\underset{m_k<j<n_{k+1}}{Max}\left|\sum_{i=m_k+1}^{j}\xi_i\right|\right)\to 0\right)=1.$$

§ 3. Let us put, for $n_k\leqq i\leqq m_k$ $(k\geqq k_0)$

(3.1) $$\xi_i'=\begin{cases}\xi_i-a_i, & if \quad |\xi_i|\leqq\dfrac{k^{1-\alpha/2}}{\log_2 k}\\ -a_i, & otherwise,\end{cases}$$

and

(3.1′) $$\xi_i''=\xi_i-\xi_i',$$

where

(3.2) $$a_i=\int_{|\xi_i|\leqq k^{1-\alpha/2}/\log_2 k}\xi_i\,dP=-\int_{|\xi_i|>k^{1-\alpha/2}/\log_2 k}\xi_i\,dP=0\left(\frac{\log_2 k}{k^{1-\alpha/2}(\log k)^{1+\varepsilon}}\right)\qquad(k\to+\infty).$$

Further let us put, for $k\geqq k_0$,

(3.3′) $$U_k=\sum_{i=n_k}^{m_k}\xi_i,$$

(3.3) $$U'_k=\sum_{i=n_k}^{m_k}\xi_i',$$

and

(3.3″) $$U''_k=\sum_{i=n_k}^{m_k}\xi_i''.$$

Then $\{\xi_i'\}$ is a sequence of m-dependent random variables having the mean value zero, and $\{U_k'\}$ is a sequence of independent random variables. The same facts are true for $\{\xi_i''\}$ and $\{U_k''\}$.

Let us put

$$W_k=\operatorname*{Max}_{n_k\leqq j\leqq m_k}\left|\sum_{i=n_k}^{j}\xi_i''\right|$$

and for $0\leqq r\leqq m$

$$W_{k,r}=\operatorname*{Max}_{0\leqq j\leqq(m_k-n_k)/(m+1)}\left|\sum_{i=0}^{m}\xi''_{n_k+i(m+1)+r}\right|.$$

Then we have

$$W_k\leqq\sum_{r=0}^{m}W_{k,r}.^{*)}$$

Since each $W_{k,r}$ is the maximum modulus of the sum of independent random variables having mean value zero, by (1.3), (3.2) and the Kolmogorov's inequality we have for any $\delta>0$

$$P\left(\frac{W_{k,r}}{\sqrt{k^\alpha\log_2 k}}>\delta\right)\leqq\sum_{i=0}^{(m_k-n_k)/(m+1)}E(\xi''_{n_k+i(m+1)+r})/k^\alpha\log_2 k$$
$$=0\left(\frac{(m_k-n_k)}{\delta^2 k^\alpha\log_2 k(\log k)^{1+\varepsilon}}\right)$$
$$=0\left(\frac{1}{\delta^2 k(\log k)^{1+\varepsilon}\log_2 k}\right)\qquad(k\to+\infty).$$

Hence, we have

$$\sum_{k=k_0}^{\infty}P\left(\frac{W_k}{\sqrt{k^\alpha\log_2 k}}>\delta(m+1)\right)\leqq\sum_{k=k_0}^{\infty}P\left(\bigcup_{r=0}^{m}\left(\frac{W_{k,r}}{\sqrt{k^\alpha\log_2 k}}>\delta\right)\right)$$

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*) This inequality is proved by the following way,

$$W_k=\max_{n_k\leqq j\leqq m_k}\left|\sum_{i=n_k}^{j}\xi_i''\right|=\left|\sum_{i=n_k}^{N(\omega)}\xi_i''\right|,$$

where $n_k\leqq N(\omega)\leqq m_k$. Hence we can write W_k by

$$W_k=\left|\Sigma'\ \xi''_{n_k+(m+1)i+r}\right|$$

and Σ' is the summation over all i and r, where

$$n_k\leqq n(n_k)+(m+1)i+r\leqq N(\omega),$$

$$W_k\leqq\sum_{r=0}^{m}\max_{0\leqq j\leqq m_k-n_k)/(m+1)}\left|\sum_{i=0}^{j}\xi''_{n_k+(m+1)i+r}\right|=\sum_{r=0}^{m}W_{k,r}.$$

$$\leqq \sum_{k=k_0}^{\infty} \sum_{r=0}^{m} P\left(\frac{W_{k,r}}{\sqrt{k^\alpha \log_2 k}} > \delta \right) = 0(1) \qquad (k \to +\infty).$$

By the Borel-Cantelli lemma, it follows that

$$P\left(\frac{W_k}{\sqrt{k^\alpha \log_2 k}} \to 0 \right) = 1, \tag{3.4}$$

On the other hand, we have

$$E(U_k''^2) = E\left(\left(\sum_{i=n_k}^{m_k} \xi_i'' \right)^2 \right) = \sum_{i=n_k}^{m_k} \sum_{\substack{j=i-m \\ n_k \leq j \leq m_k}}^{i+m} E(\xi_i'' \xi_j'') \tag{3.5}$$

$$\leqq \sum_{i=n_k}^{m_k} \sum_{j=i-m}^{i+m} \{E(\xi_i''^2)\}^{1/2} \{E(\xi_i''^2)\}^{1/2} = 0\left(k^{\alpha-1} \frac{1}{(\log k)^{1+\varepsilon}} \right) \qquad (k \to +\infty).$$

From this we have

$$\sum_k \frac{E(U_k''^2)}{k^\alpha \log_2 k} < +\infty,$$

and this implies,

$$P\left(\sum_k \frac{U_k''}{\sqrt{k^\alpha \log_2 k}} \text{ converges} \right) = 1.$$

Hence we have

$$P\left(\frac{1}{\sqrt{k^\alpha \log_2 k}} \sum_{i=k_0}^{k} U_i'' \to 0 \right) = 1. \tag{3.6}$$

From (3.4) and (3.6), we obtain

$$P\left(\frac{1}{\sqrt{k^\alpha \log_2 k}} \left(\sum_{i=k_0}^{k-1} U_i'' + \operatorname{Max}_{n_k \leq j \leq m_k} \left| \sum_{i=n_k}^{j} \xi_i'' \right| \right) \to 0 \right) = 1. \tag{3.7}$$

§ 4. we obtain from the definitions of U_i, U_i' and U_i''

$$E\left(\left(\sum_{i=k_0}^{k} U_i' \right)^2 \right) = E\left(\left\{ \sum_{i=k_0}^{k} (U_i - U_i'') \right\}^2 \right)$$

$$= E\left(\left(\sum_{i=k_0}^{k} U_i \right)^2 - 2 \sum_{i=k_0}^{k} E(U_i U_i'') + \sum_{i=k_0}^{k} \left(E(U_i'')^2 \right) \right).$$

By the m-dependences of $\{\xi_i\}$ and $\{\xi_i''\}$, we have

$$\left| E(U_i U_i'') \right| \leqq \sum_{i=n_k}^{m_k} \sum_{j=i-m}^{i+m} E(|\xi_i \xi_j|) = 0\left(\frac{k^{\alpha-1}}{(\log k)^{(1+\varepsilon)/2}} \right) \qquad (k \to +\infty).$$

Therefore we obtain by (3.5) and the above inequality,

$$E\left(\left(\sum_{i=k_0}^{k} U_i' \right)^2 \right) = E\left(\left(\sum_{i=k_0}^{k} U_i \right)^2 \right) + o(k^\alpha) \qquad (k \to +\infty).$$

By the same way, it is seen that

$$E(S_{n_{k+1}-1}^2) = E\left(\left(\sum_{i=k_0}^{k} U_i + \sum_{i=k_0}^{k} V_i + S_{n_{k_0}-1} \right)^2 \right)$$

$$= E\left(\left(\sum_{i=k_0}^{k} U_i \right)^2 \right) + 2E\left(\sum_{i=k_0}^{k} U_i (V_i + V_{i-1}) \right) + E\left(\sum_{i=k_0}^{k} V_i^2 \right)$$

$$+ 2E(U_{k_0} S_{n_{k_0}-1}) + E(S_{n_{k_0}-1}^2).$$

$$= E\left(\left(\sum_{i=k_0}^{k} U_i \right)^2 \right) + 0(k) + 0(k) + 0(1)$$

$$= E\left(\left(\sum_{i=k_0}^{k} U_i \right)^2 \right) + o(k^\alpha) \qquad (k \to +\infty).$$

If we put

$$D_k=E\left(\left(\sum_{i=k_0}^{k} U_i'\right)^2\right),$$

then we have

(4.1) $$D_k=E\left((S_{n_{k-1}}-1)^2\right)+o(k^{\alpha}) \geqq \sigma^2 k^{\alpha}/2.$$

By the definitions of ξ_i' and U_k', it is seen that

(4.2) $$\frac{U_k'}{\sqrt{2D_k/\log_2 D_k}} \leqq \frac{\sum_{i=n_k}^{m_k} |\xi_k'-a_i| + \sum_{i=n_k}^{m_k} |a_i|}{\sqrt{2D_k/\log_2 D_k}}$$

$$=O\left(\frac{k^{\alpha-1} k^{1-\alpha/2}/\log_2 k}{\sqrt{k^{\alpha}/\log_2 k}}\right)=O\left(\left(\frac{1}{\log_2 k}\right)^{1/2}\right)=o(1) \qquad (k\to+\infty).$$

Hence we apply the theorem of A. N. Kolmogorov to $\{U_k'\}$ and we obtain

(4.3) $$P\left(\overline{\lim_{n\to\infty}} \frac{\sum_{i=k_0}^{k} U_i}{\sqrt{2D_k \log_2 D_k}}=1\right)=1.$$

On the other hand we have for n $(n_k<n<n_{k+1})$

$$|E(S_{n_k}^2-E(S_n^2)| \leqq 2|E(S_{n_k}(S_n-S_{n_k}))|+E\left((S_n-S_{n_k})^2\right)$$

$$\leqq 2\sum_{i=n_k-m}^{n_k} \sum_{j=n_k}^{n_k+m} E(|\xi_i \xi_j|)+\sum_{i=n_k}^{m_k} \sum_{j=-m}^{m} E(|\xi_i \xi_{i+j}|)$$

$$=o(k^{\alpha-1})=o(k^{\alpha}) \qquad (k\to+\infty).$$

Hence we have

(4.4) $$B_n=B_{n_{k+1}-1}+o(k^{\alpha})=D_k+o(k^{\alpha}) \qquad (k\to+\infty).$$

By (1.7), (2.4) and (4.4), it is seen that for $m_k<n<n_{k+1}$

$$P\left(\overline{\lim_{n\to\infty}} \frac{\sum_{i=k_0}^{k} V_i+\sum_{i=m_k+1}^{m} \xi_i+S_{n_{k_0}-1}}{\sqrt{2B_n \log_2 B_n}}=0\right)=1.$$

By the definition of ξ_i', we have for $n_k \leqq n \leqq m_k$

$$\frac{\sum_{i=n+1}^{m_k} |\xi_i'|}{\sqrt{2B_n \log_2 B_n}}=o\left(\left(\frac{1}{\log_2 k}\right)^{3/2}\right)=o(1) \qquad (n\to+\infty),$$

hence

$$P\left(\overline{\lim_{n\to\infty}} \frac{\sum_{i=k_0}^{k} U_i'-\sum_{i=n+1}^{m_k} \xi_i'}{\sqrt{2B_n \log_2 B_n}}=1\right)$$

$$=P\left(\overline{\lim_{n\to\infty}} \frac{\sum_{i=k_0}^{k} U_i'}{\sqrt{2B_n \log_2 B_n}}=1\right)$$

$$=P\left(\overline{\lim_{n\to\infty}} \frac{\sum_{i=k_0}^{k} U_i'}{\sqrt{2D_k \log_2 D_k}}=1\right)=1.$$

On the otherhand by (3.7), for $n_k \leqq n \leqq m_k$

$$P\left(\overline{\lim_{n\to\infty}} \frac{\sum_{i=1}^{k-1} U_i''+\sum_{i=n_k}^{n} \xi_i''}{\sqrt{2B_n \log_2 B_n}}=0\right)=1.$$

And thus we obtain, by (4.4)

$$P\left(\overline{\lim_{n\to\infty}} \frac{\sum_{i=1}^{n} \xi_i}{\sqrt{2B_n \log_2 B_n}} = 1\right) = 1.$$

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