The Science Reports of the Kanazawa University Vol. IV. No. I, pp. 29—34, October, 1955.

On the Convergence of Some Random Riemann-Sums

By

Shigeru TAKAHASHI

(Received August 9, 1955)

§ 1. Let $\{t_i(\omega)\}$, $i=1,2\ldots$, be a sequence of independent random variables in a probability space (Ω, B, P) and each $t_i(\omega)$ have the uniform distribution in $[0, 1]$, that is, for $0\leqq x\leqq 1$

(1.1) $$F(x)=P(t_i(\omega)<x)=x.$$

For each ω, let $t_i^{(n)}(\omega)$ denote the i-th value of $\{t_j(\omega)\}$ $(1\leqq j\leqq n)$ arranged in the increasing order and let, for each n

(1.2) $$t_{n+1}^{(n)}(\omega)\equiv 1, \qquad t_0^{(n)}(\omega)\equiv 0.$$

Further let $f(t)$ $(-\infty<t<+\infty)$ be a Borel-measurable function with period 1 and belong to $L(0,1)$.

Then we consider the following two random Riemann-sums

(1.3) $$S_n(\omega)=\sum_{i=1}^{n} f(t_i^{(n)}(\omega))(t_{i+1}^{(n)}(\omega)-t_i^{(n)}(\omega)),$$

and

(1.4) $$S_n(\omega,s)=\sum_{i=1}^{n} f(t_i^{(n)}(\omega)+s)(t_{i+1}^{(n)}(\omega)-t_i^{(n)}(\omega)).$$

In this note we disscuss the convergences, in probability or with probability 1, of avove two Riemann-sums to $\int_0^1 f(t)dt$ and prove

Theorem 1. If $f(t)\in L(0, 1)$, then we have

$$S_n(\omega)\to\int_0^1 f(t)dt, \quad \text{in probability}, \quad \text{as } n\to+\infty.$$

Theorem 2. Let $f(t)\in L(0, 1)$ and, for an $\varepsilon>0$,

(1.5) $$\int_0^1 |f(t+h)-f(t)|\,dt=O(1/|\log|h||^{1+\varepsilon}) \qquad (h\to 0).$$

Then we have, for almost all s,

(1.6) $$P(S_n(\omega, s)\to\int_0^1 f(t)dt)=1.$$

Remark. The ω -set on which $S_n(\omega, s)\to\int_0^1 f(t)dt$ depends on s.

§ 2. Let us put, for $i\leqq n$ and $j\leqq n$ $(n=1,2,\ldots.)$,

(2.1) $$d_{i,n}(\omega)=t_{j+1}^{(n)}(\omega)-t_i(\omega), \quad \text{if} \quad t_i(\omega)=t_j^{(n)}(\omega) \quad (j=1,2,\ldots n),$$

(2.2) $$d'_{i,n}(\omega)=t_i(\omega)-t_{j-1}^{(n)}(\omega), \quad \text{if} \quad t_i(\omega)=t_j^{(n)}(\omega) \quad (j=1,2,\ldots n),$$

and

(2.3) $$d_n(\omega)=1-t_n^{(n)}(\omega).$$

Then we can write

(2.4) $$S_n(\omega)=\sum_{i=1}^{n}f(t_i(\omega))d_{i,n}(\omega)$$

(2.5) $$S_n(\omega,s)=\sum_{i=1}^{n}f(t_i(\omega)+s)d_{i,n}(\omega),$$

and

(2.6) $$\int_0^1 f(t)dt=\sum_{i=1}^{n}\int_0^{d_{i,n}(\omega)}f(t_i(\omega)+u)du+\int_0^{t_1^{(n)}(\omega)}f(u)du.$$

Lemma 1. We have, for $0\leqq h\leqq 1$,

(2.7) $$P(t_1^{(n)}(\omega)<h)=P(d_n(\omega)<h)=1-(1-h)^n$$

and

(2.8) $$P(d_{i,n}(\omega)<h)=P(d'_{i,n}(\omega)<h)=1-(1-h)^n.$$

Proof. We have, by (1.1) and the independency of $\{t_i(\omega)\}$,

$$P(t_1^{(n)}(\omega)<h)=P(\bigcup_{i=1}^{n}(t_i(\omega)<h))$$
$$=1-P(\bigcap_{i=1}^{n}(t_i(\omega)\geqq 1-h))=1-(1-h)^n.$$

By the definition of $d_{i,n}(\omega)$, we have

$$P(d_{i,n}(\omega)<h)=P([d_{i,n}(\omega)<h]\cap[t_i(\omega)\leqq 1-h])+P(t_i(\omega)>1-h)$$
$$=\int_0^{1-h}P(d_{i,n}(\omega)<h\,|\,t_i(\omega)=x)dF(x)+h,$$

where $P(E\,|\,F)$ denotes the conditional probability of E under the hypothesis F. From the independency of $\{t_i(\omega)\}$, it follows that

$$P(d_{i,n}(\omega)<h\,|\,t_i(\omega)=x)=P(\bigcup_{\substack{j=1\\ j\neq i}}^{n}(x\leqq t_j(\omega)<x+h)\,|\,t_i(\omega)=x)$$
$$=P(\bigcup_{\substack{j=1\\ j\neq i}}^{n}(x\leqq t_j(\omega)<x+h))$$
$$=1-P(\bigcap_{\substack{j=1\\ j\neq i}}^{n}(t_i(\omega)\overline{\in}[x,x+h)))$$
$$=1-(1-h)^{n-1}.$$

Hence we have

$$P(d_{i,n}(\omega)<h)=\int_0^{1-h}\{1-(1-h)^{n-1}\}\,dF(x)+h=1-(1-h)^n.$$

In the same way, we can prove the lemma for $d_n(\omega)$ and $d'_{i,n}(\omega)$.

Lemma 2. For every positive numbers x and y such that $x+y<1$, we have

$$P[(t_i(\omega)<x)\cap(d_{i,n}(\omega)<y)]=x\{1-(1-y)^{n-1}\}.$$

Proof. We have, in the same way as the poof of (2.8),

$$P[(t_i(\omega)<x)\cap(d_{i,n}(\omega)<y)]=\int_0^x P(d_{i,n}(\omega)<y\mid t_i(\omega)=z)dF(z)$$

$$=\int_0^x\{1-(1-y)^{n-1}\}dz=x\{1-(1-y)^{n-1}\}.$$

The following lemma is well known.

Lemma 3. *If* $f(t)$ belongs to $L(0,1)$ and periodic with period 1, then

$$\omega(h,f)=\underset{o<|d|\leqq h}{Max}\int_0^1|f(t+d)-f(t)|\,dt=o(1) \qquad (h\to 0).$$

§ 3. **Proof of Theorem 1.** From (2.4) and (2.6), it follows that

$$(3.1)\qquad M_n\equiv\int_\Omega\Big|S_n(\omega)-\int_0^1 f(t)dt\Big|\,dP$$

$$\leqq\sum_{i=1}^n\int_{[t_i(\omega)=t_n^{(n)}(\omega)]}dP\int_0^{d_{i,n}(\omega)}|f(t_i(\omega))-f(t_i(\omega)+u)|\,du$$

$$+\sum_{i=1}^n\int_{[t_i(\omega)\neq t_n^{(n)}(\omega)]}dP\int_0^{d_{i,n}(\omega)}|f(t_i(\omega)-f(t_i(\omega)+u)|\,du+\int_\Omega dP\int_0^{t_1^{(n)}(\omega)}|f(u)|\,du$$

$$\equiv I_{n,1}+I_{n,2}+I_{n,3}.$$

By the definitions of $t_n^{(n)}(\omega)$ and $d_n(\omega)$, we have

$$(3.2)\qquad I_{n,1}=\int_\Omega dP\int_0^{d_n(\omega)}|f(t_n^{(n)}(\omega))-f(t_n^{(n)}(\omega)+u)\ du$$

$$\leqq\int_\Omega dP\int_0^{d_n(\omega)}|f(1-d_n(\omega))|\,du+\int_\Omega dP\int_0^{d_n(\omega)}|f(1-u)|\,du.$$

By Lemma 1, we have

$$(3.3)\qquad I_{n,1}\leqq\int_0^1 n(1-x)^{n-1}x\,|f(1-x)|\,dx+\int_0^1 n(1-x)^{n-1}dx\int_0^x|f(1-u)|\,du$$

$$\equiv I_{n,1}'+I_{n,1}''.$$

Since $nx(1-x)^{n-1}\leqq nxe^{-(n-1)x}\leqq 1\ (0\leqq x\leqq 1)$, we have

$$(3.4)\qquad I_{n,1}'=\int_0^{2\log n/n}nx(1-x)^{n-1}|f(1-x)|\,dx+\int_{2\log n/n}^1 nx(1-x)^{n-1}|f(1-x)|\,dx$$

$$\leqq\int_0^{2\log n/n}|f(1-x)|\,dx+n(1-2\log n/n)^{n-1}\int_0^1|f(1-x)|dx=o(1)\ (n\to+\infty).$$

And we have

$$(3.5)\qquad I_{n,1}''\leqq\int_0^{2\log n/n}n(1-x)^{n-1}dx\int_0^{2\log n/n}|f(1-u)|\,du+\int_{2\log n/n}^1 n(1-x)^{n-1}dx\int_0^1|f(1-u)|\,du$$

$$\leqq \Big[-(1-x)^n\Big]_0^{2\log n/n}\int_0^{2\log n/n}|f(1-u)|\,du+\Big[-(1-x)^n\Big]_{2\log n/n}^1\cdot\int_0^1|f(1-u)|\,du=o(1)\qquad (n\to+\infty).$$

In the same way, we have

$$(3.6)\qquad I_{n,3}=o(1)\qquad (n\to+\infty).$$

By Lemma 2 and Lemma 3, we have,

$$(3.7)\qquad \int_{[t_i(\omega)\neq t_n^{(n)}(\omega)]} dP\int_0^{d_{i,n}(\omega)}|f(t_i(\omega))-f(t_i(\omega)+u)|\,du$$

$$=\int\int_{\substack{x\geqq 0,\ y\geqq 0\\ x+y<1}}\Big\{\int_0^y|f(x)-f(x+u)|\,du\Big\}\ (n-1)(1-y)^{n-2}dy\ dx$$

$$=\int_0^1(n-1)(1-y)^{n-2}dy\int_0^y du\int_0^{1-y}|f(x)-f(x+u)|\,dx$$

$$\leqq\int_0^{2\log n/n}(n-1)(1-y)^{n-2}dy\int_0^y du\int_0^1|f(x)-f(x+u)|\,dx$$

$$+\quad 2\int_{2\log n/n}^1(n-1)(1-y)^{n-2}dy\int_0^y du\int_0^1|f(x)|\,dx$$

$$\leqq\int_0^{2\log n/n}(n-1)(1-y)^{n-2}y\ \omega(y:f)dy+(2\int_0^1|f(x)|\,dx)(\int_{2\log n/n}^1(n-1)(1-y)^{n-2}dy)$$

$$\leqq\omega(2\log n/n;f)\Big[-(1-y)^n/n-(1-y)^{n-2}y\Big]_0^{2\log n/n}+(2\int_0^1|f(x)|\,dx)\Big[-(1-y)^{n-1}\Big]_{2\log n/n}^1$$

$$=o(1/n),$$

uniformly in i, for $1\leqq i\leqq n$, as $n\to+\infty$.

From (3.1)$\sim$(3.7), it follows that

$$M_n=\int_\Omega\Big|S_n(\omega)-\int_0^1 f(t)dt\Big|\,dP=o(1)\qquad (n\to+\infty).$$

Hence for any $\eta>0$, we have

$$P(|S_n(\omega)-\int_0^1 f(t)dt|>\eta)\leqq M_n/\eta=o(1)\qquad (n\to+\infty).$$

which completes the proof.

§ 4. **Proof of Theorem 2.** From Theorem 1, it is seen that for each s, we have

$$(4.1)\qquad S_n(\omega,s)\to\int_0^1 f(t)dt,\ \text{in probability, as } n\to+\infty.$$

By (2.5), we have

$$S_n(\omega,s)-S_{n-1}(\omega,s)=d_{n,n}(\omega)g_n(\omega)$$

where

$$g_n(\omega)=\begin{cases}f(t_n(\omega)+s)\ , & \text{if}\quad \omega\in[t_n(\omega)=t_1^{(n)}(\omega)],\\ f(t_n(\omega)+s)-f(t_n(\omega)-d'_{n,n}(\omega)+s), & \text{if}\quad \omega\in[t_n(\omega)\neq t_1^{(n)}(\omega)].\end{cases}$$

Let us put

$$E_1 \equiv \left[\omega;\ t_n(\omega) = t_1^{(n)}(\omega)\right]$$

$$E_2 \equiv \left[\omega;\ d_{n,n}(\omega) \geqq 2\log n/n\right]$$

$$E_3 \equiv \left[\omega;\ d'_{n,n}(\omega) \geqq 2\log n/n\right]$$

and

$$E'_i \equiv \Omega - E_i.$$

Then from the independency of $\{t_i(\omega)\}$ and (1.1), it follows that

$$(4.2) \qquad P(E_1) = 1/n$$

and, from Lemma 1,

$$(4.3) \qquad P(E_2) = P(E_3) = O(1/n^2) \qquad (n \to +\infty).$$

On the other hand we have

$$T_n \equiv \int_0^1 ds \int_\Omega |S_n(\omega, s) - S_{n-1}(\omega, s)|\, dP$$

$$= \int_0^1 ds \int_{E_1 \cap E_2} |d_{n,n}(\omega) f(t_n(\omega)+s)|\, dP + \int_0^1 ds \int_{E_1 \cap E'_2} d_{n,n}(\omega)\, |f(t_n(\omega)+s)|\, dP$$

$$+ \int_0^1 ds \int_{E'_1 \cap E_2} d_{n,n}(\omega)\, |f(t_n(\omega)+s) - f(t_n(\omega) - d'_{n,n}(\omega)+s)|\, dP$$

$$+ \int_0^1 ds \int_{E'_1 \cap E'_2 \cap E_3} d_{n,n}(\omega)\, |f(t_n(\omega)+s) - f(t_n(\omega) - d'_{n,n}(\omega)+s)|\, dP$$

$$+ \int_0^1 ds \int_{E'_1 \cap E'_2 \cap E'_3} d_{n,n}(\omega)\, |f(t_n(\omega)+s) - f(t_n(\omega) - d'_{n,n}(\omega)+s)|\, dP \equiv I_1 + I_2 + I_3 + I_4 + I_5.$$

By (4.3) we have

$$I_1 \leqq \int_{E_2} d_{n,n}(\omega)\, dP \int_0^1 |f(t_n(\omega)+s)|\, ds = O(P(E_2)) = O(1/n^2) \qquad (n \to +\infty).$$

By (4.2) we have

$$I_2 = \int_{E_1 \cap E'_2} d_{n,n}(\omega)\, dP \int_0^1 |f(t_n(\omega)+s)|\, ds = O(2\log n/n\ P(E_1)) = O(\log n/n^2) \qquad (n \to +\infty).$$

By the same way, we have

$$I_3 \leqq \int_{E_2} d_{n,n}(\omega)\, dP \int_0^1 |f(t_n(\omega)+s) - f(t_n(\omega) - d'_{n,n}(\omega)+s)|\, ds$$

$$= O(P(E_2)) = O(1/n^2) \qquad (n \to +\infty).$$

and

$$I_4 \leqq \int_{E_3} d_{n,n}(\omega)\, dP \int_0^1 |f(t_n(\omega)+s) - f(t_n(\omega) - d'_{n,n}(\omega)+s)|\, ds$$

$$= O(P(E_3)) = O(1/n^2) \qquad (n \to +\infty).$$

By (1.5) and Lemma 3, we have

$$I_5 \leqq \int_{E'_n} d_{n,n}(\omega)dP \int_0^1 |f(t_n(\omega)+s)-f(t_n(\omega)-d'_{n,n}(\omega)+s)|\,ds$$

$$\leqq \omega\ (2\log n/n:\ f) \int_{\Omega} d_{n,n}(\omega)dP = O(1/n\,(\log n)^{1+\varepsilon}) \qquad (n\to+\infty).$$

Therefore, it follows that

$$\sum_n \int_0^1 ds \int_{\Omega} |S_n(\omega, s)-S_{n-1}(\omega, s)|\,dP<+\infty.$$

This shows that, for almost all s,

$$\sum_n \int_{\Omega} |S_n(\omega, s)-S_{n-1}(\omega, s)|\,dP<+\infty$$

and this implies that, for almost all s,

$$(4.4) \qquad P(\ S_n(\omega,s)\ \ converges\ )=1.$$

(4.4) and (4.1) prove Theorem 2.

References

[1] A. Kolmogorov, Grundbegriffe der Wahrsheinlichkeits-rechnung, Berlin. [1933].

[2] D. A. Darling, On a class of problems related to the random division of an interval, Annals Math. Stat. vol. 24 (1953) pp. 239-253.

[3] S. Takahashi, Notes on the Riemann-sum, Proc. Japan Acad. vol. 31 (1955) pp. 8-13.