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Volume, Surface-Area and Total-Mean-Curvature

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1. Introduction.

The attempt of establishing the spatial isoperimetric inequality has been made in various ways as is well-known. Among these, G. Bol [3]* dealt the problem with the internal parallel surfaces of a closed convex surface. A. Renyi [11] discussed completely the problem in the Euclidean plane E_2 by improving Bol's treatment, and gave an explicit positive integral representation of the isoperimetric deficiency. In a similar view the spatial isoperimetric problem should be completely discussed by the method of internal parallel surfaces. For the purpose, we shall first give the differential and integral formulas for the enclosed volume, the surface area and the total mean curvature of a closed convex body. Our treatment is distinguished by the introduction of the form-figure and the characteristic function of a closed convex body from G. Bol's paper ([3] and [4]).

2. Surface and internal parallel surface.

2.1. Internal parallel surfaces.

The definition of a parallel surface of a given closed convex surface K can be formulated in various ways. By certain definitions, it is possible that the internal parallel surface may not be convex. So, in order to obtain always the convex internal parallel surface of K, we define it in the following manner.**

For the purpose, let us define the positive direction towards the inner side of a closed convex surface K on each surface normal. Taking a half space which is separated by a supporting plane of K and contains the corresponding positive surface normal, we define it as the positive half-space referring to the supporting plane. Now, let us move all supporting planes of K by the same distance t ($\geqq 0$) along the corresponding normals. Then, if all the positive-half spaces referring to the moved supporting planes have an intersection, we call the surface which encloses the intersection, as "*the internal parallel surface of* K *at the distance t*" and denote it with K(t). By the difinition, K(0) is K itself.

* Numbers in brackets refer to the list of references at the end of the paper.

** See [4], P 39.

2.2. Internal parallel convex polyhedra.

Let P be a convex polyhedron in E_3 and let us consider all the internal parallel polyhedra P(t) of P. For sufficiently small value of t, the number of vertices, edges and faces of P(t) is the same as those of the original polyhedron P(0) respectively. However, as the value of t increases, the numbers of these elements of P(t) decrease in general, namely, some of edges or faces may shrink to a point or points when a "*critical value*" of t is reached. Each of such critical values of t corresponds to a radius-length of one or many spheres which touch internally to four or more faces of P(0). Let us denote such values of t with ρ_i $(i=1,2,\ldots)$ in the increasing order and with r the length of the radius of the greatest inscribed sphere in P(0). The number of the greatest inscribed sphere of radius r is determined by the form of P(0). That is to say, the number is one or infinite. In the latter case, their centres stand on a line or a plane due to the convexity of P(0). Then, the interval [0, r] is divided into a finite number ot the subintervals by ρ_i $(i=1,2,\ldots)$. In any case, when the value of t starting from 0 increases, the number of vertices, edges and faces of P(t) decrease, but are constant at the the same interval of t, say $\rho_i \leqq t < \rho_{i+1}$ $(i=1,2,\ldots)$ and the form of P(t) degenerates gradually at every critical value of t. When t converges to r, P(t) converges to a point, a line-segment of finite length or a plane segment of finite area. Consequently, the internal parallel polyhedra P(t) are defined only at the interval $0 \leqq t \leqq r$ of the parameter.

3. Differential formulas.

3.1. Sequence of internal parallel polyhedra.

P(t) $(0 \leqq t \leqq r)$ is an internal parallel polyhedron to P(0). We shall study the change of the value of the enclosed volume, the surface area and the total mean curvature of P(t). We start with the enclosed volume V(t) and the surface area S(t) of P(t) as defined.

In a proper way, let us mark the edges of P(0) with numbers and keep the numbers to the corresponding edges of P(t). $l_i(t)$ is the length of the i-th edge of P(t) and $\gamma_i(t)$ its dihedral angle. Then, taking P(t) and $P(t+\varDelta t)$ which belong to the same layer of the internal parallel surfaces, the difference of their volumes V(t) and $V(t+\varDelta t)$ is expressed as follows,

$$V(t)-V(t+\varDelta t)=S(t+\varDelta t)\cdot\varDelta t-(\varDelta t)^2\cdot\sum_i l_i(t+\varDelta t)\tan\frac{\gamma_i}{2}+\frac{(\varDelta t)^2}{3}\sum_i\left\{l_i(t)-l_i(t+\varDelta t)\right\}\tan\frac{\gamma_i}{2}. \quad (1)$$

Hence,

$$V'(t)=-S(t). \quad (2)$$

Precisely, V(t) is differentiable over the interval [0, r] except for the critical values ρ_i $(i=1,2,\ldots)$ and we should take the left-side differential coefficient of V(t) at every critical value ρ_i.

Next, about the surface area S(t), we have

$$S(t)-S(t+\varDelta t)=2\sum_i l_i(t+\varDelta t)\tan\frac{\gamma_i}{2}\cdot\varDelta t-\varDelta t\sum_i\left\{l_i(t)-l_i(t+\varDelta t)\right\}\tan\frac{\gamma_i}{2}. \quad (3)$$

Hence

$$S'(t)=-2\sum_i l_i(t)\tan\frac{\gamma_i}{2}. \quad (4)$$

Prof. Blaschke [2] defined the edge curvature M of a convex polyhedron as

$$M=\frac{1}{2}\sum_i l_i\gamma_i. \quad (5)$$

Similary, let us denote the following quantity with $\bar{M}$, namely

$$\bar{M}(t)=\sum_i l_i(t)\tan\frac{\gamma_i}{2}. \quad (6)^*$$

Then, definitely

$$\bar{M}(t)\geqq M(t). \quad (7)$$

Or we have by (5) and (6)

$$\bar{M}(t)=M(t)+\sum_i l_i(t)\left(\tan\frac{\gamma_i}{2}-\frac{\gamma_i}{2}\right). \quad (8)$$

Now, the above formula can be written as follows,

$$S'(t)=-2\bar{M}(t). \quad (9)$$

In this case too, we should take the left-side differential coefficients at the critical values ρ_i $(i=1,2,\ldots)$.

Next, the derivative of $\bar{M}(t)$ is calculated as follows. First we have

$$-\frac{\bar{M}(t+\varDelta t)-\bar{M}(t)}{\varDelta t}=\sum_i\frac{l_i(t)-l_i(t+\varDelta t)}{\varDelta t}\tan\frac{\gamma_i}{2}. \quad (10)$$

The right side of (10) is equal to the surface area of a polyhedron $\Pi(t)$ which is constructed with the planes, each of which is tangent to a unitsphere and parallel to the corresponding face of the internal parallel polyhedron P(t) respectively. Now we give the following definition. *A polyhedron Π, which is circumscribed to a unit sphere and each of whose surfaces is parallel to the corresponding face of P respectively, is called the form figure*** *of the convex polyhedron* P.

Now, the right side of (10) is equal to the surface-area of the form-figure $\Pi(t)$ of P(t)***. Consequently, its value is constant at each subinterval $(\rho_i, \rho_{i+1}]$ between two adjacent critical values.

And, excepting the critical values ρ_i $(i=1,2,\ldots)$, we have

$$\bar{M}'(t)=-\kappa(t), \quad (11)$$

where

$$\kappa(t)=\sum_i\frac{dl_i(t)}{dt}\cdot\tan\frac{\gamma_i}{2}. \quad (12)$$

and we take the left-side differential coefficient of $\bar{M}(t)$ at every critical value ρ_i $(i=1,2,\ldots)$. Let us call $\kappa(t)$ *the characteristic function or characteristic of the convex polyhedron* $P(t)$.

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* G. Bol denoted $\bar{M}$ with M* [3], p. 32.

** Th. Kaluza called the corresponding figure in the plane by the same name.

***G. Bol and A. Renyi defined the characteristic function of a polygon as the double area of the form-figure in E_2. But it is better to difine it as the periphery of the form-figure in order to generalize the dimension of the space.

Hereupon, G. Bol* denotes the corresponding quantity with C*, and $\kappa(t)$ is different from the total curvature $C(t)=4\pi E(K(t))$, where $E(K(t))$ denotes the Euler characteristic of K(t). He says that it is not easy to give a simple geometrical explanation for C* and it is of no importance even if C* takes any value.

But we can summarize the above result as follows.

THEOREM 1. *The characteristic function* κ *of a convex polyhedron P is equal to the surface area of the form-figure* Π *of P.*

3.2. Characteristic function of P(t).

If the original polyhedron P(0) is circumscribed to a unit sphere of radius r, the form-figures $\Pi(t)$ of P(t) $(0\leqq t\leqq r)$ are unchangeable, so that the characteristic function $\kappa(t)$ of P(t) is constant. In general, the number of faces of $\Pi(t)$ decreases as t increases, namely the characteristic function is an increasing function. To say more precisely, if ρ_i is one of the critical values of t and $t_{i_1}<\rho_i<t_{i_2}$, the number of faces of the form-figure Π (t_{i_1}) will be identical with one of $\Pi(t_{i_2})$ or more than that. Therefore

$$\kappa(t_{i_1})\leqq\kappa(t_{i_2}),\qquad 0\leqq t_{i_1}<\rho_i<t_{i_2}\leqq r. \tag{13}$$

Or, we can state as follows.

THEOREM 2. *The characteristic function* $\kappa(t)$ $(0\leqq t\leqq r)$ *of a convex polyhedron* P(t) *is a monotone increasing function.*

Moreover, the characteristic function is a step function which is discontinuous at finite points which correspond to the critical ualues of t.

Next, comparing the surface area of the form figure $\Pi(t)$ with that of the unit sphere, we obtain the following theorem.

THEOREM 3. *The value of the characteristic function* $\kappa(t)$ *of a convex polyhedron* $P(0)$ *is greater than* 4π, *that is,*

$$\kappa(t)>4\pi,\qquad (0\leqq t\leqq r). \tag{14}$$

Moreover, $\kappa(t)$ being an increasing function, it follows by (11) that $\overline{M}(t)$ is a concave function.

On the other hand, S(t) and $\overline{M}(t)$ being non-negative functions, it follows by (2) and (9) that V(t) and S(t) are convex functions for $0\leqq t\leqq r$ [7].

4. Closed convex surface and formulas.

4.1. Approximation by convex polyhedra.

Let K be a closed convex surface with inner points in the Euclidean three-space E_3. By virtue of the theorem of choice of Blaschke [1], we can take out a partial sequence $\{P_n\}$ of convex polyhedra which converge to K. Thus we have

$$\lim_{n\to\infty} P_n=K. \tag{15}$$

Now, if the enclosed volume V_n, the surface area S_n, the total mean curvature $\overline{M}_n$ and

* See [3] p. 33 and [6] p. 31.

the characteristic function κ_n of P_n have V, S, $\overline{M}$ and κ as their limits respectively, that is,

$$\lim_{n\to\infty} V_n = V, \qquad \lim_{n\to\infty} S_n = S, \qquad \lim_{n\to\infty} \overline{M}_n = \overline{M}, \qquad \lim_{n\to\infty} \kappa_n = \kappa, \tag{16}$$

we define V, S, $\overline{M}$ and κ as the enclosed volume, the surface area, the total mean curvature and the characteristic function of K, respectively.

In like manner, the inner parallel surface K(t) of K can be approximated by a sequence $\{P_n(t)\}$ of convex polyhedra. At this time, the supremum r of the parameter t is equal to the radius of the greatest sphere which is inscribed to the original surface K(0).

For the purpose of establishing the differential formulas of V(t), S(t), $\overline{M}$(t) of K(t), let us quote the following lemma of F. Riesz [12]:

If a sequence of convex functions converges to a limit function, the derivatives of the functions of the sequence converge to the derivative of the limit function, provided that latter exist.

Since $V_n(t)$, $S_n(t)$ of $P_n(t)$ are convex functions and $\overline{M}(t)$ a concave function in [0, r] (§ 3.2.), we can obtain the following theorem.

THEOREM 4. *If $V(t)$, $S(t)$, $\overline{M}(t)$ and $\kappa(t)$ are the enclosed volume, the surface area, the total mean curvature and the characteristic function of an inner parallel surface* K(*t*) *of the closed convex surface* K(0), *we have*

$$V'(t) = -S(t), \tag{17}$$

$$S'(t) = -2\overline{M}(t), \tag{18*}$$

$$\overline{M}'(t) = -\kappa(t) \tag{19}$$

excepting the critical values in the interval [0, r] *and we take the left-side differential coefficients of them respectively at the critical values.*

4.2. Charateristic function of K(t).

When a closed convex surface K(t) be approximated by $\{P_n(t)\}$, the form figure Π(t) of K(t) should be defined as the limit figure of the form figure $\Pi_n(t)$. Namely, Π(t) is circumscribed to a unit sphere and each supporting plane of Π(t) is parallel to a supporting plane of K(t). Hence, we have the following theorem owing to Theorem 1.

THEOREM 5. *The characteristic function $\kappa(t)$ of a closed convex surface* K(*t*) *is, in value, equal to the surface area of the form-figure $\Pi(t)$ of* K(*t*) *for* $0 \leq t \leq r$.

Now, if a closed convex surface K is composed entirely of regular points, its form-figure Π coincides with the unit sphere ([5], pp. 13–14), and its characteristic κ is equal to 4π. But so far as K has singular points, that is, angular points, or edges, the form-figure Π has singular points corresponding to them, so that the characteristic κ is greater than 4π.

* $\overline{M} = M + \sum_i \int \left(\tan\frac{\gamma_i}{2} - \frac{\gamma_i}{2}\right) dl_i$, where the integrations are extended over all edges of K. $\overline{M}$ is identical with M for the surface without edge.

The form figure $\Pi(t)$ of K(t) in $0 \leqq t \leqq r$ changes its form at every critical value of t which is equal to one of the principal radii of normal curvature, and the characteristic function $\kappa(t)$ corresponds to the change by increasing its value. Hence, referring to Theorem 2 in case of P(t), we have:

Theorem 6. *The characteristic function $\kappa(t)$ of a closed convex surface* K(*t*) *is a monotone increasing function for* $0 \leqq t \leqq r$.

We can cite following examples of such surfaces whose characteristic functions are constantly equal to 4π for $0 \leqq t \leqq r$;

(i) A sphere,

(ii) A convex hull of two spheres w'th different centres,*

(iii) A convex hull of a torus, and so forth.

On the other hand, we can give the following examples of surfaces with the property that the characteristic function is constant for $0 \leqq t \leqq r$;

(iv) A sphere with a hood or hoods**,

(v) A general cone with a plane end which is bounded by a convex curve,

(vi) A cylinder with two plane ends which are bounded by the same convex curve, and so forth.

The above justifies the following theorem corresponding to Theorem 3.

Theorem 7. *If $\kappa(t)$ is a characteristic function of a cloced convex surface* K(*t*), *we have*

$$\kappa(t) \geqq 4\pi, \qquad 0 \leqq t \leqq r. \tag{20}$$

5. Integral formulas.

5.1. A classification of closed convex surfaces.

In case of the closed convex surfaces, the limit figure of the internal parallel surface K(t) of K(0) as $t \rightarrow r$ is a point, a line-segment of a finite length or a plane-segment of a finite area. Now, we give the following definition ([4], p. 40).

Definition. The limit-figure of the internal parallel surface K(t) $(t \rightarrow r)$ *is called a kernel of the original surface* K. *According as the kernel of* K *is a point, a line-segment or a plane-segment, we call it a point-kernel, a line-kernel or a plane-kernel.*

Then it is easy to find that the surface with a line-kernel is partially composed of a cylindrical surface and the surface with a plane-kernel is partially composed of two parallel plane segments whose distance is equal to 2r.

For example, let K be a cylinder of revolution whose radius be r and height 2h; then, if h is equal to, greater or less than r, the kernel of K is a point, a line-segment of length 2(h-r) or a plane-segment of area $\pi(r-h)^2$.

* It is admitted for the convex hull constructed by three or more congruent spheres with different centres in a plane.

**The number of hoods is not restricted to one provided the sphere with hoods retains the convexity of the surface.

5.2. Integral formulas (I).

(i) Taking the formula (19), we have:

$$\overline{M}(t)=\int_t^r \kappa(s)ds+\overline{M}(r), \qquad 0\leqq t\leqq r, \tag{21}$$

where $\overline{M}(r)$ denotes $\lim_{t\to r}\overline{M}(t)$. In fact, we can find the value of $\overline{M}(r)$ as follows:

Case I. The surface with a point-kernel; $\overline{M}(r)=0$,

Case II. The surface with a line-kernel; $\overline{M}(r)=l.\sigma(r)$, where l denotes the length of the kernel and $\sigma(r)$ the limiting value $(t\to r)$ of the area $\sigma(t)$ of the form figure* of the plane curve along which the cylindrical part of the original surface is cut by a plane at right angles to the kernel.

Case III. The surface with a plane-kernel; $\overline{M}(r)=2U$, where U denotes the length of the perimeter of the kernel.

(ii) Next, by (18), we have:

$$S(t) = \int_t^r 2\overline{M}(s)ds+S(r), \qquad 0\leqq t\leqq r, \tag{22}$$

where $S(r)$ is zero for surfaces with a point-, or line-kernel and in case of surfaces with a plane-kernel, it is equal to the double area 2F of the plane-kernel of K(0).

(iii) Finally, we obtain by (17),

$$V(t) = \int_t^r S(s)ds, \qquad 0\leqq t\leqq r. \tag{23}$$

5.3. Integral formulas (II).

We now write down the above formulas under the three cases of surfaces as follows.

Case I. Surface with a point-kernel;

$$\begin{cases} \overline{M}(t) = \int_t^r \kappa(s)ds, & (21_1) \\ S(t) = 2\int_t^r \overline{M}(s)ds, & (22_1) \\ V(t) = \int_t^r S(s)ds. & (23_1) \end{cases}$$

Case II. Surface with a line-kernel;

$$\begin{cases} \overline{M}(t) = \int_t^r \kappa(s)ds+l.\sigma(r), & (21_2) \\ S(t) = 2\int_t^r \overline{M}(s)ds, & (22_2) \\ V(t) = \int_t^r S(s)ds, & (23_2) \end{cases}$$

where l is the length of the line-kernl and $\sigma(r)$ the plane area of the form-figure corresponding to the cylindrical part of K(t) $(t\to r)$.

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* We mean by it the same figure as Th. Kalza defined in the plane; see the foot note at p. 3.

Case III. Surface with a plane-kernel;

$$\begin{cases} \overline{M}(t) = \int_t^r \kappa(s)ds + 2U, & (21_3) \\ S(t) = 2\int_t^r \overline{M}(s)ds + 2F, & (22_3) \\ V(t) = \int_t^r S(s)ds, & (23_3) \end{cases}$$

where U is the length of the perimeter of the plane kernel and F the plane area of the kernel.

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