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On the Burg's Numbers

By

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1. R. Burg found the numbers N to base 10 such that the numbers obtained by reversing its digits are multiple yN of N, in particular for $y=9$, 4. [1], [2].

We shall research these numbers N and y to base g.

Throughout this note all small Latin letters denote non negative integers.

2. N may be expressed uniquely in the form

$$N= x_1 g^{n-1}+\cdots\cdots+ x_n,$$

where every x is one of $0, 1, 2,\cdots\cdots, g-1$ and x_1 is not 0.

We abbreviate this expression to

$$N=\{x_1, x_2,\cdots\cdots, x_n\}, (g),$$

the representation of N in decimal in the scale of g.

We have to consider the $n+1$ equations

(1) $$s_{k+1}+x_k y=x_{n-k}+s_k g, \ (0\leqq k\leqq n),$$

where

$$g\geqq 3, \ 2\leqq y\leqq g-1, \ 0\leqq x_k \leqq g-1, \ (0\leqq k\leqq n), \ x_0 \neq 0, \ 0\leqq s_k \leqq y-1, \ 0\leqq k\leqq n$$

and

$$s_0 =s_{n+1}=0$$

Furthermore, we assume that $s_1 = 0$.

3. As we have ($k=0$ and n in (1)), $x_0 y=x_n$ and $x_n y=x_0 +s_n y$, we get $x_0 (y^2 -1) \equiv 0 \pmod g$. Since $(x_0, g)=d$ is the highest common divisor of x_0 and g, we can put $x_0 =x^*_0 d$, $g=g^*d$, where, $(x^*_0 g^*)=1$.

Thus we have $y^2 \equiv 1 \mod g^*$. The solutions of this congruence are given by $y = ig^* \pm 1$ $(1\leqq i\leqq d)$. Since $x_n = x_0 y=x^*_0 (g^*-1)$, $d\leqq g-1$, we have $x^*_0 \leqq (g-1)/(g^*-1)d=(g^*d-1)/(g^*-1)d=1+(d-1)/(g^*-1)d<2$. Thus we get $x^*_0 =1$, $d=x_0$ and

(2) $$g=x_0 (y+1).$$

We have by (1), that

$$\begin{cases} x_k y=s_{k+1}=x_{n-k}+s_k g \\ x_{n-k} y+s_{n-k+1} =x_k +s_{n-k} g. \end{cases}$$

From these equations and $y+1 \mid g$, (g is divisible by $y+1$), we have

$$x_k = x_{n-k}+x_0 (s_k - s_{n-k}) + \frac{1}{y+1}(s_{n-k+1} -s_{k+1})$$

Since x_k and $x_{n-k}+x_0 (s_k -s_{n-k})$ are integers, $(s_{n-k+1} -s_{k+1})/(y-1)$ is an integer.

Since $0 \leqq s_{n-k+1}/(y+1)<1$, we have $-1<(s_{n-k+1}-s_{k+1})/(y+1) \leqq 0$, hence $(s_{n-k+1}-s_{k+1})/(y+1)=0$. Thus, it follows that

$$(3) \qquad \begin{cases} x_k = x_{n-k}+x_0\,(s_k - s_{n-k}) \\ s_k = s_{n-k+2} \end{cases}$$

From (1) and (3), we have

$$(4) \qquad \begin{cases} x_k = x_0\, y\omega_k + x_0\, \omega_{n-k} - \omega_{n-k+1} \\ \omega_k = \omega_{n-k+2} \end{cases}$$

where $\omega_k = s_k\,/\,(y-1)$ and $0 \leqq \omega_i \leqq 1$, $2 \leqq i \leqq n$. From (1) $(k=n)$ and (2), we have $s_n\, y = x_n\, y - x_0 = x_0\,(y^2-1)=g(y-1)$ and we get $\omega_n = \omega_2 = y-1$ (integer). We prove by induction that all of ω are integers.

We assume that $\omega_k = \omega_{n-k+2}$, $2 \leqq k \leqq j+2$ are integers. First we prove by induction that $x_0^p\, \omega_{n-j-p} = x_0^p\, \omega_{j+p+2}$ $(0 \leqq p)$ are integers.

From (4) $(k=j+p)$, we have $x_0^p x_{j+p} - x_0^p y\, \omega_{j+p} + x_0^p \omega_{n-j-p} - x_0^{p-1} \omega_{n-j-(p-1)}$.

From this equation and our induction hypothesis, we get our desired result. Next, we assume $n=2m$, then from (4), $x_0^{m-j-2} x_m = x_0^{m-j-1} y\omega_m + x_0^{m-j-1}\omega_m - x_0^{m-j-2}\omega_{m+1}$. Since $x_0^{m-j-1}\omega_m$ is an integer, $x_0^{m-j-2}\omega_m$ is also an integer. Thus we have also by induction that ω_{n-j-1} is an integer.

Similarly we can prove our result for $n=2m+1$.

We have from (4) and the above

$$(5) \qquad \begin{cases} x_k = x_0\, y\, \omega_k + x_0\, \omega_{k+2} - \omega_{k+1}, \\ \omega_k = \omega_{n-k+2} = 1 \text{ or } 0, \quad (0 \leqq k \leqq n). \end{cases}$$

$$(2) \qquad g = x_0\,(y+1)$$

Thus we obtain the Table I, and we know that all x_k can take only 6 values. x_0 and y are decided by (2).

ω_k	ω_{k+2}	ω_{k+1}	x_k
1	1	1	$g-1$
1	0	1	$x_0\, y-1$
0	1	1	x_0-1
1	0	0	$x_0\, y$
0	1	0	x_0
0	0	0	0

Table I.

If we take one of these 6 values for x_k, then x_{k-1} and x_{k+1} are given by the Table II.

x_{k-1}	x_k	x_{k+1}
0, yx_0	0	0, x_0
0, yx_0	x_0	x_0-1
x_0	x_0-1	$g-1$, yx_0-1
x_0-1, $g-1$	$x_0\, y-1$	yx_0
yx_0-1,	$x_0\, y$	0, x_0
$g-1$	$g-1$	yx_0-1, $g-1$

Table II.

Conclusion. If $s_1 = 0$ we have

$N = \{x_0, x_0 - 1, \underbrace{g - 1, \ldots, g - 1}_{t}, x_0 y - 1, x_0 y\} \cdot y\ (g),\ (t \geqq 0)$ and their trivial connections.

We call these numbers "Burg's Numbers" to base g.

4. If $s_1 = s \neq 0$, we have

$$g = x_0 (y + e),$$

where

$$e^2 - se - 1 \equiv 0 \ (mod\ \ y + e).$$

Thus we have

$$\left(\frac{4+e^2}{p}\right) = 1, \quad p \mid g,$$

where $\left(\frac{x}{p}\right)$ denotes the Legendre's symbol.

We shall reserve it for another occasion to treat this case, and only two examples are shown as follows :

1) $(1, 2m, \ldots, 2m, 2m-1) \cdot m$, $(2m+1)$

2) $(1, 7, \ldots, 7, 6) \cdot 5$; $(1, 8, 12, \ldots, 12, 13, 6) \cdot 5$; $(1, 9, 19, 6) \cdot 5$ (29) etc.,

5. Remark 1. If we write the number of "Burg's Number" (n digits) with ζ_n, then we can put $\zeta_2 = \zeta_3 = 0$ and $\zeta_1 = 1$. If we define $\zeta_0 = 1$ and put $\zeta_{2n} = \zeta_{2n+1} = \alpha_n$, then we have the recurring formula of Fibonacci numbers ([1] and [2], Chap. XIX) :

$$\alpha_n = \alpha_{n-1} + \alpha_{n-2},$$

Remark 2. For $g=10$, we have only two factorizations of 10 as follows :

$$10 = 1 \cdot 10 = 2 \cdot 5.$$

In this case, it is easy to see that $s=0$. For, we have the following :

	$x_0 = 1,\ g^* = 10$				$x_0 = 2,\ g^* = 5$			
e	1	2	3	4	1	2	3	4
s	0,5	4	1,6	0,5	0	4	1	0
y	9, –	8	7, –	6, –	4	3	2	1
$x_n \leqq 9$	9, –	–	7, –	6, –	8	–	5	–
$x_n y \equiv x_0$ (10)	1, –	–	–, –	–, –	2	–	–	–

Table III.

Thus we have the Burg's Numbers to base 10 as follows [1] .

$(1,0,9,\ldots,9,8,9) \cdot 9, (10)$ and $(2,1,9,\ldots,9,7,8) \cdot 4, (10)$.

References

[1], R. Burg : Welche Zahlen gehen in umkehrter Ziffernfolge ein Vielfaches ihres Wertes ?, Sitz. Ber. Math. Gesell. 1916.

[2], L. E. Dickson : History of the theory of Numbers, Vol. **1**, Chelsea, 1952.