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On the Series of Some Independent Random Variables.

By

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1. Let $\{X_n\}$ be a sequence of independent random variables with a common distribution function $F(x)$ and $\{a_n\}$ be a decreasing sequence of positive numbers. Further let us put

$$P_0 = Prob.\left[\sum_n |a_n X_n| < +\infty\right]$$

$$P_1 = Prob.\left[\sum_n a_n X_n \ converges\right]$$

and

$$P_2 = Prob.\left[\sum_n a_n \left(X_n - \int_{|a_n x| \leq 1} x\, dF(x)\right) \ converges\right].$$

It is seen that each of

(1.1) $$P'_0 = Prob.\left[\overline{lim}_{n\to\infty}\ a_n \sum_{i=1}^{n} |X_i| \leq 1\right] = 1,$$

and

(1.2) $$P'_1 = Prob.\left[\overline{lim}_{n\to\infty}\ \left|a_n \sum_{i=1}^{n} X_i\right| \leq ½\right] = 1$$

implies

(1.3) $$Prob.\left[\overline{lim}_{n\to\infty}\ |a_n X_n| \leq 1\right] = 1$$

and therefore, by the Borel-Cantelli lemma, (1.1) and also (1.2) imply

(1.4) $$\sum_n Prob.\left[|X_n| > \frac{1}{a_n}\right] = \sum_n \int_{|x|>\frac{1}{a_n}} dF(x) < +\infty.$$

In this note we prove that under some conditions on $\{a_n\}$, (1.4) implies $P_i = 1$ $(i=0, 1, 2)$. Especially, by Knopp's theorem in the theory of series, $P_0 = 1$ (or $P_1 = 1$) implies $P'_0 = 1$ (or $P'_1 = 1$).

2. THEOREM 1. If for some $\lambda > 1$

(2.1) $$\frac{a_{n+1}}{a_n} \leq \left(\frac{n}{n+1}\right)^{\lambda} \qquad n = 1, 2, \ldots,$$

then (1.4) implies $P_0 = 1$.

PROOF. By the three series theorem, it is sufficient to show that

$$\sum_n V_n = \sum_n \left[\int_{|a_n x| \leq 1} (a_n x)^2\, dF(x) - \left(\int_{|a_n x| \leq 1} |a_n x|\, dF(x)\right)^2\right] < +\infty$$

and

$$\sum_n \overline{M}_n = \sum_n \int_{|a_n x| \leq 1} |a_n x| \, dF(x) < +\infty .$$

On the other hand, it may be seen that

$$0 \leq V_n \leq \int_{|a_n x| \leq 1} |a_n x|^2 dF(x) \leq \overline{M}_n$$

and hence, it is sufficient to prove that $\sum_n \overline{M}_n < +\infty$.

We have

$$\sum_{n=1}^{\infty} \int_{|a_n x| \leq 1} |a_n x| \, dF(x) = \sum_{n=1}^{\infty} a_n \sum_{m=1}^{n} \int_{\frac{1}{a_{m-1}} < |x| \leq \frac{1}{a_m}} |x| \, dF(x)$$

$$\leq \sum_{n=1}^{\infty} a_n \sum_{m=1}^{n} \frac{1}{a_m} \int_{\frac{1}{a_{m-1}} < |x| \leq \frac{1}{a_m}} dF(x) = \sum_{m=1}^{\infty} \frac{1}{a_m} \int_{\frac{1}{a_{m-1}} < |x| \leq \frac{1}{a_m}} dF(x) \sum_{n=m}^{\infty} a_n ,$$

where

$$\frac{1}{a_0} \equiv 0.$$

From (2.1), we have

$$a_n \leq a_m \left(\frac{m}{n} \right)^{\lambda} \qquad \text{for } n \geq m ,$$

and we have

$$\sum_{n=m}^{\infty} a_n \leq a_m m^{\lambda} \sum_{n=m}^{\infty} \left(\frac{1}{n} \right)^{\lambda} \leq A \, a_m m$$

where A is a constant independent of m.

Therefore, it follows that

$$\sum_{n=1}^{\infty} \overline{M}_n \leq A \sum_{m=1}^{\infty} m \int_{\frac{1}{a_{m-1}} < |x| \leq \frac{1}{a_m}} dF(x)$$

$$= A \sum_{m=1}^{\infty} \left[\int_{|x| > \frac{1}{a_m}} dF(x) + \int_{\frac{1}{a_{m-1}} < |x| \leq \frac{1}{a_m}} dF(x) \right] < +\infty .$$

THEOREM 2. Let $F(x)$ be a symmetric distribution function and for some $\lambda > \frac{1}{2}$

(2.2) $$\frac{a_{n+1}}{a_n} \leq \left(\frac{n}{n+1} \right)^{\lambda} \qquad n = 1, 2, \ldots .$$

Then (1.4) implies $P_1 = 1$.

PROOF. It is sufficient to show that

$$\sum_n \int_{|a_n x| \leq 1} (a_n x)^2 \, dF(x) < +\infty$$

and this can be proved by the same way as the proof of $\sum_n \overline{M}_n < +\infty$.

THEOREM 3. If for some $\lambda > \frac{1}{2}$

(2.3) $$\frac{a_{n+1}}{a_n} \leq \left(\frac{n}{n+1} \right)^{\lambda} \qquad n = 1, 2, \ldots ,$$

then (1.4) implies $P_2 = 1$.

PROOF. Let us define random variables Y_n and Z_n as follows.

$$Y_n = \begin{cases} 0, & \text{if} \quad |X_n| > \dfrac{1}{a_n} \\ X_n, & \text{if} \quad |X_n| \leqq \dfrac{1}{a_n} \end{cases}$$

and

$$Z_n = \begin{cases} X_n, & \text{if} \quad |X_n| > \dfrac{1}{a_n} \\ 0, & \text{if} \quad |X_n| \leqq \dfrac{1}{a_n} \end{cases} \qquad n = 1, 2, \ldots .$$

From (1.4) and the above definitions, it follows that

$$\text{(2.4)} \qquad Prob.\left[\sum_n |a_n Z_n| < +\infty\right] = 1,$$

and

$$D_n = E\left[(a_n Y_n)^2\right] - \left[E(a_n Y_n)\right]^2 = \int_{|a_n x| \leqq 1} (a_n x)^2 \, dF(x) - \left(\int_{|a_n x| \leqq 1} a_n x \, dF(x)\right)^2$$

$$M_n = E(a_n Y_n) = \int_{|a_n x| \leqq 1} a_n x \, dF(x).$$

As the proof of Theorem 2, we can show that

$$0 \leqq \sum_n D_n \leqq \sum_n \int_{|a_n x| \leqq 1} (a_n x)^2 \, dF(x) < +\infty.$$

On the other hand, the random variable $(a_n Y_n - M_n)$ has the mean value zero and the dispersion D_n. Hence, by Khintchine-Kolmogoroff's theorem, we have

$$\text{(2.5)} \qquad Prob.\left[\sum_n (a_n Y_n - M_n) \text{ converges}\right] = 1.$$

From (2.4) and (2.5), we get the required result.

3. 1°. Suppose that

$$\text{(3.1)} \qquad Prob.\left[\sum_n \left|\frac{1}{n} X_n\right| < +\infty\right] = 1.$$

Then, by the Borel-Cantelli lemma, it follows that for any constant $c > 0$,

$$\sum_n Prob.\left[|X_n| > cn\right] < +\infty.$$

Therefore, we have

$$\int_{-\infty}^{+\infty} |x| \, dF(x) < +\infty \quad ,$$

and the strong law of large numbers shows that

$$Prob.\left[\frac{1}{n} \sum_{k=1}^{n} |X_k| \to \int_{-\infty}^{+\infty} |x| \, dF(x)\right] = 1.$$

On the other hand, from (3.1), we have

$$Prob.\left[\frac{1}{n} \sum_{k=1}^{n} |X_k| \to 0\right] = 1.$$

Hence, we have

$$\int_{-\infty}^{+\infty} |x|\, dF(x) = 0$$

and this is equivalent to

(3.2) $$Prob.\left[X_n = 0\right] = 1 \qquad n = 1, 2, \ldots .$$

Therefore, λ in Theorem 1 can not be replaced by 1 except a trivial case (3.2).

2°. Let $F(x)$ be symmetric and

(3.3) $$Prob.\left(\sum_n \frac{1}{\sqrt{n}} X_n \ \textit{converges}\right) = 1.$$

Then we have, by the same way as 1°,

$$\int_{-\infty}^{+\infty} x^2\, dF(x) < +\infty ,$$

and from the central limit theorem and (3.3), we have

$$\int_{-\infty}^{+\infty} x^2\, dF(x) = 0.$$

This shows that (3.3) implies (3.2). Therefore, λ in Theorem 2 can not be replaced by ½ except a trivial case (3.2).

3°. Let

$$Prob.\left[\sum_n \frac{1}{\sqrt{n}} \left(X_n - \int_{|x| \leqq \sqrt{n}} x dF(x)\right) \ \textit{converges}\right] = 1$$

and $\overline{X}_n$ denote the symmetrized random variable of X_n .
Then we have

$$Prob.\left[\sum_n \frac{1}{\sqrt{n}} \overline{X}_n \ \textit{converges}\right] = 1.$$

Therefore, by the discussion in 2°, we have

$$Prob.\left[\overline{X}_n = 0\right] = 1 \qquad n=1, 2, \ldots .$$

This shows that

(3.4) $$Prob.\left[X_n = m\right] = 1 \qquad n = 1, 2, \ldots .$$

where m is a constant independent of n.
Hence λ in Theorem 3 can not be replaced by ½ except a trivial case (3.4).