The Science Reports of the Kanazawa University, Vol. III, No. 2, pp. 199—207, July, 1955

On Mean Values and Geometrical Probabilities in E_n.

By

Shigeru OSHIO

(Received February 3, 1955)

Introduction. In this report V.3, pp.35-43, we treated the subject in the Euclidean three-space E_3 [5].* The aim of the present paper is to generalize my previous result [5] in an Euclidean n-space E_n. For the purpose, we shall first define the uniform-figure-systems in E_n in the same manner as befor, and discuss the subject under classified cases.

§ 1. Definition

1.1. Covering of Euclidean Space E_n.

Let σ_0 be a closed hypersurface in the Euclidean n-space E_n. Let us define a discontinuous rigid motion T_1 which operates upon σ_0 and satisfies with the following condition (α). Let us begin with marking σ_0 at its initial position in E_n. If we operate T_1 upon σ_0, σ_0 will move to the next position. Again, let us mark σ_0 at such a position.

Let us set the following condition (α), that is ; (α) *The two adjacent surfaces have a part of them in common but neither of them has any inner point of the domain enclosed by the other surface in common.* Let us operate T_1 and T_1^{-1} upon σ_0 in the same direction**) i_1-times in all, then we obtain a tubiformsurface σ_1 which is joined by i_1-pieces of σ_0 piece by piece. Subsequently, let us define another rigid motion T_2 which operates upon σ_1 in a direction which is independent of that of T_1 and satisfies with the condition (α). Again, operating T_2 and T_2^{-1} upon σ_1 i_2-times in all, we obtain a leaf-like hypersurface which is composed of i_2-pieces of σ_1 piece by piece. Let us denote this leaf-like hypersurface by σ_2.

In such a manner, let us suppose that the construction, by means of which a series $\{\sigma_k\}$ of hypersurfaces and a series $\{T_k\}$ of discontinuous rigid motions of which T_k operates upon σ_{k-1} i_k-times in common with T_k^{-1} and satisfies the condition (α), are defined, in succession, $k=1,2,\ldots,$ n, and a closed hypersurface σ_n is obtained at last. Then σ_n will be composed of $i_1 i_2 \ldots i_n$-pieces of surfaces congruent to σ_0, or in other words, the whole domain which is enclosed by σ_n is divided into $i_1 i_2 \ldots i_n$-pieces each of which is congruent to the domain enclosed by σ_0 without overlapping or without

*) Number in brackets refer to the bibliography at the end of the paper.

**) It can be proved that any given rigid motion in E_n is the resultant of a uniquely defined rotation and a uniquely defined translation along the axis of the rotation.

leaving gaps. After this construction, if we increase i_1, i_2, ... i_{n-1} and i_n to infinity simultaneously, the whole space E_n is covered by ∞^n-pieces of domains, each being congruent to the domain enclosed by σ_0, without overlapping or without leaving gaps. Now let us call such a construction" a covering of E_n by σ_0", the domain ϑ enclosed by σ_0, „a fundamental cell" in the covering and each domain which is arranged by the covering, „ a unit cell" in the covering.

1.2. Uniform Figure Systems

Connecting with the above covering of E_n by σ_0, let us define the uniform arrangements of figures. For the purpose, let us prepare the following figure-set:

(k) A set $\mathfrak{S}^k = \{f_j^k\}$ of k-dimensional manifolds $f_j^k (j = 1, 2, \ldots, p), p < \infty$ and the sum*) of k-dimensional measure of $f_j^k (j=1, 2, \ldots, p)$ is finite.

Explaining in detail, the meaning of the above definition is as follows. For $k=0$, f_j^0 reduces a point and the sum of zero-dimensional measure of f_j^0's is equal to the number of these points, namely p. For $k=1$, f_j^1 reduces a curve and the one-dimensional measure of f_j^1 is equal to its curve length, and so forth. Now, if we cover E_n by a fundamental cell σ_0 attached with a set $\mathfrak{S}^k$ of k-dimensional manifolds, every unit cell will be alloted with a figure set $\mathfrak{S}^k$.

At this time, let us select the figure set $\mathfrak{S}^k$ in the covering of E_n by σ_0 with $\mathfrak{S}^k$, (β) *every figure set at all unit cells may not intersect with any set at another cells.* Now, in such a covering of E_n by σ_0 attached with a figure set $\mathfrak{S}^k$, let us call an original set $\mathfrak{S}^k$ of k-dimensional manifolds, by the name of „the fundamental figure" and every one alloted to each unit cell, „the unit figure." In consequence, we shall obtain such a uniform arrangement of the unit figure alloted to each unit cell in whole space E_n.

In the end, let us cross out all the covering hypersurfaces, then we obtain a uniform arrangement of a unique figure set $\mathfrak{S}^k$ in E_n. Let us classify such a uniform arrangement of σ_k in E^n corresponding to the kind of the fundamental figure (k) by calling „uniform-k-dimensional manifold-system in E_n". Hence,

(0) a uniform-point-system,
(1) a uniform-curve-system,
(2) a uniform-surface-system,
... ...
(k) a uniform- $\mathfrak{S}^k$ -systen
... ...
(n-1) a uniform-hypersurface-system.

§ 2. Uniform-Point-System in E_n

2.1 Set of points of a finite number. Let $\mathfrak{S}^0$ be a set $\{P_j\}$ of points P_i ($i=1, 2, \ldots, \rho$) which are fixed in E_n and $\mathfrak{R}$ a closed hypersurface whose n-dimensional enclosed volume

*) We shall speak it by calling k-dimensional measure of in the following.

is V and which moves in E_n .

Then, using the kinematic density, by W. Blaschke [1]

$$\dot{\mathfrak{K}} = \prod_{i=1}^{n} \dot{s}_i \prod_{j=1}^{n-1} \dot{\omega}_j \ , \tag{1}$$

we can obtain the following formula, which represents the total measure of positions of moving $\mathfrak{K}$ by which a fixed point P is contained, that is,

$$\int_{p \subset \mathfrak{K}} \dot{\mathfrak{K}} = \Omega_n \cdot V, \tag{2}$$

where

$$\Omega_n = \omega_1 \, \omega_2 \cdots \omega_{n-1} = \frac{2^{n-1} \pi^{\frac{1}{4}(n+2)(n-1)}}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n-1}{2}\right)\cdots\Gamma\left(\frac{2}{2}\right)} \ , \quad \omega_i = \frac{2\pi^{\frac{i+1}{2}}}{\Gamma\left(\frac{i+1}{2}\right)} \tag{3}$$

Then the total measure of the positions of $\mathfrak{K}$ which contains some points of $\mathfrak{S}$ is given by the formula

$$\int_{\mathfrak{S} \cap \mathfrak{K} \neq 0} m\dot{\mathfrak{K}} = \rho \Omega_n V, \tag{4}$$

where m represents the number of the points of $\mathfrak{S}$ contained in $\mathfrak{K}$ at a position and the integration is extended all over the intersection points of $\mathfrak{S}$ and $\mathfrak{K}$.

2.2 Uniform-point-system and moving surface.

(1). Let us take a uniform-point-system in E_n which is defined by the fundamental cell attached with ρ-points and move a closed hypersurface $\mathfrak{K}$ whose n-dimensional volume is V.

Let us estimate the mean value of the number of points belonging to the system which are contained by $\mathfrak{K}$ at a position.

Let us proceed our treatment comparing with my preuious paper [5] in E_3 .

(2) In the same manner with that in the case of E_3 , let us begin with the definition of the domain $\mathfrak{A}$ which consists of $\mu_1 \mu_2 \ldots \mu_n$-pieces of unit-cell. Here μ_i-pieces are lying along the direction corresponding to T_i . Then, we obtain a point-set $\mathfrak{P}_{\mathfrak{A}}$ belonging to the system which is contained in the domain $\mathfrak{A}$ and consists of $\rho \prod_{i=1}^{n} \mu_i$ points.

Hereupon, defining two domains $\overline{\mathfrak{A}}$ and $\overline{\overline{\mathfrak{A}}}$ after the analogy to the case E_3, we can obtain the superset $\mathfrak{P}_{\overline{\mathfrak{A}}}$ and the subset $\mathfrak{P}_{\overline{\mathfrak{A}}-\overline{\overline{\mathfrak{A}}}}$ to the set $\mathfrak{P}_{\mathfrak{A}}$. Here, $\mathfrak{P}_{\overline{\mathfrak{A}}}$ consists of $\rho \prod_{i=1}^{u} (\mu_i + 2\nu_i)$-points of the system and $\mathfrak{P}_{\overline{\mathfrak{A}}-\overline{\overline{\mathfrak{A}}}}$, $\rho \{\prod_{i=1}^{n} (\mu_i + 2\nu_i) - \prod_{i=1}^{n} (\mu_i - 2\nu_i)$-points of the system.

(3) Under these preparations, let us estimate the mean value of the number of points of the uniform-point-system which are contained by a moving closed hypersurface $\mathfrak{K}$. First, applying (4) for the point-set $\mathfrak{P}_{\overline{\mathfrak{A}}}$, we obtain

$$\int_{\mathfrak{P}_{\overline{\mathfrak{A}}} \cap \mathfrak{K} \neq 0} m \, \dot{\mathfrak{K}} = \rho \Omega_n V \prod_{i=1}^{n} (\mu_i + 2\nu_i) \tag{5}$$

where ρ denotes the number of points of the system in a unit cell ϑ, V the enclosed volume by $\mathfrak{K}$ and Ω_n is given by (3).

If we divide the integral at the left side of (5) into two parts $I_{P \in \mathfrak{A}}$ and $I_{P \bar{\in} \mathfrak{A}}$: $I_{P \in \mathfrak{A}}$ is such a total measure of the position of $\mathfrak{K}$ as the origin P attached to $\mathfrak{K}$ is contained in $\mathfrak{A}$ and $I_{P\bar{\in}\mathfrak{A}}$ is the rest of the integral. Then, we can rewrite (5), as follows,

$$I_{P\in\mathfrak{A}} + I_{P\bar{\in}\mathfrak{A}} = \rho\Omega_n V \prod_{i=1}^{n} (\mu_i + 2\nu_i). \tag{6}$$

In the same way as befor, if we represent by $\int_{P\in\vartheta} m\dot{\mathfrak{K}}$ the total measure of positions of $\mathfrak{K}$ so that P attached to $\mathfrak{K}$ may be contained in a unit cell ϑ and $\mathfrak{K}$ may contain $m(\geqq 1)$ -points of the system, we can express $I_{P\in\mathfrak{A}}$, as follows,

$$I_{P\in\mathfrak{A}} = \prod_{i=1}^{n} \mu_i \int_{P\in\vartheta} m\dot{\mathfrak{K}} \tag{7}$$

On the same account as before, we can put

$$I_{P\bar{\in}\mathfrak{A}} \leqq \rho\Omega_n V\{\prod_{i=1}^{n} (\mu_i + 2\nu_i) - \prod_{i=1}^{n} (\mu_i - 2\nu_i)\},$$

and therefore

$$\lim_{\mu_i\to\infty} \frac{I_{P\bar{\in}\mathfrak{A}}}{\prod_{i=1}^{n} \mu_i} = 0 \tag{8}$$

Now, if we divide the both members of (6) by $\prod_{i=1}^{n} \mu_i$ and take account of (7) and (8), we obtain the following equation as the limit equation as $\mu_i \to \infty$ $(i=1, 2, \ldots, n)$

$$\int_{P\in\vartheta} m\dot{\mathfrak{K}} = \rho\Omega_n V. \tag{9}$$

On the other hand, the tatal measure of $\mathfrak{K}$ at such positions P attached to $\mathfrak{K}$ is contained in a unit cell ϑ is given as follows,

$$\int_{P\in\vartheta} \dot{\mathfrak{K}} = \Omega_n C^{*} \tag{10}$$

where C denotes the volume of a unit cell ϑ.

Hence, dividing (9) by (10), we obtain the following theorem.

THEOREM 1. *When a closed hypersurface $\mathfrak{K}$ of the n-dimensional volume* V *moves in E_n where a uniform-point-system is defined, the mean value $\bar{m}$ of the number of points of the system contained in $\mathfrak{K}$, is given by*

$$\bar{m} = \rho \frac{V}{C}, \tag{11}$$

where ρ denotes the number of points belonging to a unit cell and the volume of a unit cell.

It is a matter of course that the above theorem comprises the corresponding cases of $n=2$ [3], or 3 [5] as its special case, and $\bar{m}$ is given by the same expression independently of the dimension of the space.

Consequently, in the special case in which the number of points contained by $\mathfrak{K}$ under these configuration, is exclusively limited to m_1 or m_2 $(m_1 > m_2 \geqq 0)$, the probability P_1 for the case that the number in question is m_1 is given $P_1 = \frac{\rho V - Cm_2}{C(m_1 - m_2)}$ in every

*) In order to save trouble of reexplanation of the same procedure, we shall speak of the equation (10) as „a limit equation by covering procedure of the whole space E_n with the equation (4)".

number of dimension.

§ 3. Uniform-$\mathfrak{S}^k$-System and Moving Manifold $\mathfrak{R}^r$.

3.1. Mean value of the measure $S_{k+r-n}(\mathfrak{S}^k \cap \mathfrak{R}^r)$.

(1) In E_n Let us define a uniform-$\mathfrak{S}^k$-system which is constructed by the fundamental figure-set $\mathfrak{S}^k$ of k-dimensional manifolds f_j^k ($j=1,2,\dots, p$), and move a r-dimensional manifold $\mathfrak{R}^r$ whose r-dimensional volume is finite. Now, we assume that $r+k-n \geqq 0$ and let $\mathfrak{S}^k \cap \mathfrak{R}^r$ be the $(r+k-n)$-dimensional manifolds of the intersection of some of $\mathfrak{S}^k$'s with $\mathfrak{R}^r$ and $S_{k+r-n}(\mathfrak{S}^k \cap \mathfrak{R}^r)$ $(k+r-n)$-dimensional volume of the intersection $\mathfrak{S}^k \cap \mathfrak{R}^r$. Here, when $r+k-n=0$, $S_0(\mathfrak{S}^k \cap \mathfrak{R}^r)$ denotes the number of intersection points of $\mathfrak{S}^k \cap \mathfrak{R}^r$. Then let us estimate the mean value $S_{k+r-n}(\mathfrak{S}^k \cap \mathfrak{R}^r)$ of $S_{k+r-n}(\mathfrak{S}^k \cap \mathfrak{R}^r)$.

(2) At this time, we have a satisfactory formula [4] by L. A. Santaló. Adjusting it to the present case, we can express it as follows. Let C^k be a fixed k-dimensional manifold and $\mathfrak{R}^r$ a moving r-dimensional manifold in E^n. If $r+k-n \geqq 0$, we have

$$\int_{C_k \cap \mathfrak{R}^r \neq 0} S_{k+r-n}(C^k \cap \mathfrak{R}^r)\,\dot{\mathfrak{R}} = \frac{\Omega_{n+1}\omega_{r+k-n}}{\omega_k \quad \omega_r} S_k(C^k)\, S_r(\mathfrak{R}^r), \tag{12}$$

where

$$\Omega_{n+1} = \omega_1\, \omega_2 \cdots \omega_{n-1}\omega_n, \qquad \omega_i = \frac{2\pi^{\frac{i+1}{2}}}{\Gamma\left(\frac{i+1}{2}\right)}.$$

Further, let us use the following formula by L. A. Santaló [4] which gives expression to the total measure of positions of $\mathfrak{R}^r$ which leave a point of $\mathfrak{R}^r$ invariant. That is,

$$\int_{Total} d\,\mathfrak{R}^{r[P]} = \Omega_n. \tag{13}$$

(3) Let us find the mean value of $S_{k+r-n}(\mathfrak{S}^k \cap \mathfrak{R}^r)$. In the covered space E_n, let us take a domain $\mathfrak{A}$ which contains $\prod_{i=1}^{n} \mu_i$-pieces of unit cell ϑ and a figure-set $\mathfrak{S}^k_{\mathfrak{A}}$ which is consisted of $\prod_{i=1}^{n} \mu_i$-pieces of unit figure $\mathfrak{S}^k$. And then two domains $\overline{\mathfrak{A}}$ and $\overline{\overline{\mathfrak{A}}}$, or a figuresuperset $\mathfrak{S}^k_{\overline{\mathfrak{A}}}$ and a subset $\mathfrak{S}^k_{\overline{\mathfrak{A}}-\overline{\overline{\mathfrak{A}}}}$ are determined with reference to $\mathfrak{R}^r$. Then, using (12), we have

$$\int_{\mathfrak{S}^k_{\overline{\mathfrak{A}}} \cap \mathfrak{R}^r \neq 0} S_{k+r-n}(\mathfrak{S}^k_{\overline{\mathfrak{A}}} \cap \mathfrak{R}^r)\,\dot{\mathfrak{R}} = \frac{\Omega_{n+1}\omega_{k+r-n}}{\omega_k\, \omega_r} S_k(\mathfrak{S}^k)\, S_r(\mathfrak{R}^r) \prod_{i=1}^{n}(\mu_i + 2\nu_i). \tag{14}$$

Now, in the same maner as in §2. 2, (2), let us denote by $\int_{P\in\vartheta} S_{k+r-n}(\mathfrak{S}^k_{\overline{\mathfrak{A}}} \cap \mathfrak{R}^r)\,\dot{\mathfrak{R}}$ such a total measure of positions of $\mathfrak{R}^r$ as the origin P of the moving frame attached to $\mathfrak{R}^r$ is contained in a unit cell ϑ and $\mathfrak{R}^r$ has common points with the unit figure $\mathfrak{S}^k$. Taking the same procedure as in § 2,2. (2), we can obtain the limit equation by the covering procedure of the whole space E_n with the above foxmula (14), as follows,

$$\int_{P\in\vartheta} S_{k+r-n}(\mathfrak{S}^k \cap \mathfrak{K}^r)\,\dot{\mathfrak{K}} = \frac{\Omega_{n+1}\,\omega_{k+r-n}}{\omega_k\,\omega_r}\, S_k(\mathfrak{S}^k)\, S_r(\mathfrak{K}^r). \tag{15}$$

On the other hand, the total measure of $\mathfrak{K}^r$ at sucha position as P attached to $\mathfrak{K}^r$ is contained in a unit cell of the system is given using (13), as follows,

$$\int_{p\in\vartheta} \dot{\mathfrak{K}} = \Omega_n C \tag{16}$$

where C denotes the n-dimensional volume of a unit cell. Dividing (15) with (16), we have the following theorem.

THEOREM 2. *When a r-dimensional manifold $\mathfrak{K}^r$ whose r-dimensional volume is finite moves in E_n where a uniform-k-dimensional manifold $\mathfrak{S}^k$-system is defined, the mean value $\overline{S_{k+r-n}(\mathfrak{S}^k \cap \mathfrak{K}^r)}$ of the (k+r−n)-dimensional volume $S_{k+r-n}(\mathfrak{S}^k \cap \mathfrak{K}^r)$ of the intersection $\mathfrak{S}^k \cap \mathfrak{K}^r$ is given by*

$$\overline{S_{k+r-n}(\mathfrak{S}^k \cap \mathfrak{K}^r)} = \frac{\omega_n\,\omega_{r+k-n}}{\omega_k\,\omega_r}\cdot\frac{S_k(\mathfrak{S}^k)\,S_r(\mathfrak{K}^r)}{C}, \tag{17}$$

where $S_k(\mathfrak{S}^k)$ is the sum of k-dimensional volume of the figure set belonging to $\mathfrak{S}^k$ and S_r the r-dimensional volume of $\mathfrak{K}$ and C the n-dimensional volume of a unit cell.

Consequently, it has been definitely showned by the formula (17) that the above mean value is the same with one in the case that the uniform-figure-system is constructed by $\mathfrak{K}^r$ as the fundamental figure and $\mathfrak{S}^k$ is employed for the moving figure.

3.2 Special cases of the theorem (2).

(1) Case (1) $k=1$, $r=n-1$

Now, $r+k-n=0$, so that $S_{k+r-n}(\mathfrak{S}^k \cap \mathfrak{K}^r) = S_0(\mathfrak{S}^1 \cap \mathfrak{K}^{n-1})$ represents the number m of intersection-points of $\mathfrak{S}^1$ with $\mathfrak{K}^{n-1}$.

Therfore

$$\overline{m} = \overline{S_0(\mathfrak{S}^1 \cap \mathfrak{K}^{n-1})} = \frac{\omega_n\,\omega_0}{\omega_1\,\omega_{n-1}}\cdot\frac{L F_{n-1}}{C} = \frac{\Gamma\left(\frac{n}{1}\right)}{\Gamma\left(\frac{n+1}{2}\right)}\frac{LF_{n-1}}{\Pi^{\frac{1}{2}} C}. \tag{18}$$

The formula gives the mean value of the number of the intersection- points of the moving hypersurface $\mathfrak{S}$ of the $(n-1)$-dimensional volume F_{n-1} with the curves of the uniform-curve $\mathfrak{S}^1$-system defined by a curve-set of length L.

For $n=1$, $\mathfrak{S}^0$ reduces a set $\{P_i\}$ of some points, say $i=p$ and $\mathfrak{S}^1$ reduces a set of line-segments. Then m denotes the number of the points of $\mathfrak{K}$ $(P_1, P_2, \ldots P_p)$ which stand on the set $\mathfrak{S}^1$ at a moment. According to (18), we have $\overline{m} = \dfrac{pL}{C}$.

For $n=2$, $\mathfrak{K}^1$ reduces a set of curves of the total length l. Then, the mean value $\overline{m}$ of the number of intesection point of with the uniform curve-system in the plane amount to $\dfrac{2lL}{\pi C}$.

This is in accord with the result in the plane by L. A. Santaló [3]. For $n=3$, $\mathfrak{K}^2$ reduces a set of surfaces of the total surface area F, the mean value $\overline{m}$ of the intersection points of $\mathfrak{K}^2$ with the curves of the system is given by (18), as follows,

$$\bar{m}=\frac{LE}{2C}.$$

The result is identical with that which we obtained in the previous paper [5].

Case (II) $k=2$, $r=n-1$.

In the present case, $k+r-n=1$, so that $S_{k+r-n}(\mathfrak{S}^k \cap \mathfrak{K}^r)=S_1(\mathfrak{S}^2 \cap \mathfrak{K}^{n-1})$ represents curve length of intersections of $\mathfrak{S}^2$ with $\mathfrak{K}^{n-1}$.

Therefore

$$\bar{s}=\overline{S_1(\mathfrak{S}^2 \cap \mathfrak{K}^{n-1})}=\frac{\omega_n\,\omega_1}{\omega_2\,\omega_{n-1}}\ \frac{S_2(\mathfrak{S}^2)\,S_{n-1}(\mathfrak{K}^{n-1})}{C},$$

or

$$\bar{s}=\frac{\pi^{\frac{1}{2}}\,\Gamma\left(\frac{n}{2}\right)}{2\Gamma\left(\frac{n+1}{2}\right)}\ \frac{F\cdot f_{n-1}}{C}, \tag{19}$$

where F is $S_2(\mathfrak{S}^2)$, precisely, the total surface area of $\mathfrak{S}^2$ and $f_{n=1}$ is the $(n-1)$-dimensional volume of $\mathfrak{K}^{n-1}$.

For $n=2$, $\mathfrak{K}^{n-1}=\mathfrak{K}^1$ reduces a set of curves of the total curve-length l. Then $\mathfrak{S}^2$ is a set of closed curves of the total plane area F. For s which is the mean value of the length of the curve segments which are included by the closed curves belong ing to the uniform-$\mathfrak{S}^2$-system, we have

$$\bar{s}=\frac{F\cdot l}{C}. \tag{20}$$

For $n=3$, $\mathfrak{K}^{n-1}=\mathfrak{K}^2$ reduces a set of surfaces of the total surface area f, $\mathfrak{S}^2$ is a set of surfaces of the total surface area F. In the present case, denotes the mean value of the length of the intersection-curves of moving $\mathfrak{K}^2$ with the uniform-surface $\mathfrak{S}^2$-system. Then, we have

$$\bar{s}=\frac{\pi\cdot F\cdot f}{4C}. \tag{21}$$

This is the same with the result which we obtained in the previous paper [5].

4. Geometrical Probabilities.

4.1. Kinematic formula and geometrical probabilies.

Let us suppose that a uniform-curve-system is defined with a fundamental cell of n-dimensional volume and a fundamental curve-set of the total length L in E_n. Now, taking a convex hypersurface $\mathfrak{K}$ of the $(n-1)$-dimensional volume f_{n-1} which encloses a n-dimensional manifold $\mathfrak{G}_1$ of the volume V, let $\mathfrak{K}$ move in the space E_n. Then, we can classify $\mathfrak{K}$'s position with reference to the uniform-curve-system into three cases as follows;

(i) $\mathfrak{K}$ includes completely one *or* any pieces of curves belonging to the uniform-curve-system,

(ii) $\mathfrak{K}$ has common points with the uniform-curve-system,

(iii) $\mathfrak{G}_1$ has no common point with the uniform-curve-system.

Let us estimate the probabilities P_i $(i=1, 2, 3)$ for one of the above classified cases which may occur. For the purpose, we have the following kinematic formula (2) in the euclidean space E_n by S.S.Chern,

$$\int K(D_0 D_1)\Sigma_1 = J_n\{M^{(0)}_{n-1}V+M^{(1)}_{n-1}V_0+\frac{1}{n}\sum_{k=0}^{n-2}\binom{n}{k+1}M^{(0)}_{k}M^{(1)}_{n-2-k}\}, \quad (22)$$

where $K(D_0\cdot D_1)=I_{n-1}X(D_0\cdot D_1)$.

In our case, we have following values for fundamental curve-set as the fixed fiqure Σ_0;

$$W_0=0,\ M^{(0)}_i=0\ (i=1, 2, \ldots, n-3) \qquad M^{(0)}_{n-2}=\frac{\omega_{n-2}L}{n-1}, \qquad M^{(0)}_{n-1}=\omega_{n-1},$$

and for the closed hypersurface $\mathfrak{K}$ as the moving figure Σ_1; $V_1=V$, $M_0^{(1)}=f_{n-1}$.
If we denote the number of the intersection points of $\mathfrak{K}$ with the curves of the system by m in general, we have

$$K(D_0\cdot D_1)=m\,\omega_{n-1}.$$

Moreover, according to our symbol, J_n equals to Ω_n. Then, the kinematic formula(22) reduces as follows,

$$\int_{\mathfrak{S}_1\cap\mathfrak{S}^1\neq 0} m\dot{\mathfrak{K}}=\Omega_n V+\frac{\Omega_{n-1}\ \omega_{n-2}}{n-1}\cdot L\cdot f_{n-1}. \quad (23)$$

Now, let us suppose to take a domain $\mathfrak{A}$ in the space E^n covered with the uniform-curve-system and denote the curve-set of the system in $\mathfrak{A}$ by $\mathfrak{S}^1_{\mathfrak{A}}$, then using (23), we have

$$\int_{\mathfrak{S}_1\cap\mathfrak{S}^1_{\mathfrak{A}}\neq 0} m\dot{\mathfrak{K}}=(\Omega_n V+\frac{\Omega_{n-1}\ \omega_{n-2}}{n-1}\cdot L\cdot f_{n-1})\prod_i(\mu_i+2\nu_i). \quad (24)$$

In the same manner as in § 2.2, (2), denoting by $\int_{P\in\vartheta} m\dot{\mathfrak{K}}$ such a total measure of positions of $\mathfrak{K}$ as the origin P attached to $\mathfrak{K}$ is contained in a unit cell ϑ and $\mathfrak{S}_1$ has common points with some of the curves in ϑ and taking the covering procedure of the whole space E_n with (24), we have the following limit equation,

$$\int_{P\in\vartheta} m\dot{\mathfrak{K}}=\Omega_n V+\frac{\Omega_{n-1}.\ \omega_{n-2}}{n-1}L\cdot f_{n-1}. \quad (25)$$

Hereupon, we can divide the integral standing at the left side of (25) into tow parts, that is

$$\int_{\mathfrak{C}\subset\mathfrak{S}_1}\dot{\mathfrak{K}}+\int_{\mathfrak{S}1\cap\mathfrak{K}\neq 0} m\dot{\mathfrak{K}}=\Omega_n V+\frac{\Omega_{n-1}.\ \omega_{n-2}}{n-1}L\cdot f_{n-1},$$

where $\mathfrak{C}$ represents any pieces of curves belonging to a unit cell.

On the other hand, the integral $\int_{\mathfrak{S}'\cap\mathfrak{K}\neq 0} m\dot{\mathfrak{K}}$ can be written by (15) as follows,

$$\int_{\mathfrak{S}^1\cap\mathfrak{K}\neq 0} m\dot{\mathfrak{K}}=\frac{\Omega_{n+1}\ \omega_0}{\omega_{n-1}\ \omega_1}\cdot L\cdot f_{n-1}, \quad (26)$$

$$\int_{\mathfrak{C}\subset\mathfrak{S}_1}\dot{\mathfrak{K}}=\Omega_n V+\left(\frac{\Omega_{n-1}.\ \omega_{n-2}}{n-1}-\frac{\Omega_{n+1}\cdot\omega_0}{\omega_{n-1}\cdot\omega_1}\right)L\cdot f_{n-1} \quad (27)$$

Then, using (10), (26) and (27), we have

$$\begin{cases} p_1 = \dfrac{V}{C} + \left(\dfrac{\Omega_{n-1} \cdot \omega_{n-1}}{n-1} - \dfrac{\Omega_{n+1} \cdot \omega_0}{\omega_{n-1} \cdot \omega_1} \right) \dfrac{L \cdot f_{n-1}}{C}, \\ p_2 = \dfrac{\omega_n \cdot \omega_0}{\omega_{n-1}\, \omega_1} \dfrac{L \cdot f_{n-1}}{c}, \\ p_3 = 1 - (p_1 + p_2). \end{cases} \tag{28}$$

For example, for $n=3$, we have

$$p_1 = \frac{4V - L f_2}{4C} \quad p_2 = \frac{L f_2}{2C}, \quad p_3 = \frac{4(C-V) - L f_2}{4C}.$$

The result is identical with the one which we obtained the previous paper in the space E_3 .

References

(1) W. Blaschke, Ermittlung der Dichten fuer lineare Unterräume im E_n . Actualites scientifiques et industrielles 252, Paris 1935, Hermann & Cie.

(2) S. S. Chern, On the kinematic formula in the euclidean space of n-dimensions, Ame. J. Math., vol.**74** (1952) pp.227-236.

(3) L. A. Santaló, Sobre valores medios y probabilidades geometricas, Abhandlungen aus dem mathematischen Seminar der Hansischen Univ. Bd.**13** (1940) pp.284-294.

(4) L. A. Santaló, Geometria Integral en Espacios de Curvatura Constante, Publicacions de la Comision Nacional de la Energia Atomica, Serie Matematica, vol. **1**-N^0 1 Buenos Aires, 1952.

(5) S. Oshio, On mean values and geometrical probabilities in E_3, Science Reports of the Kanazawa University, Vol.**111** (1955) pp.35-43.