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Note on the Subregions Bounded by the Level Curves of the Green's Function

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In this paper we shall extend *Walsh's* theorem ([1]) in §§ **1, 2** and investigate the linearlization of $T(w)$ with property T in a domain D in § **3** (see § **1** for definition) and prove the existence of non-linear one-to-one conformal mapping of D onto itself in § **4.**

§**1.** On the extension of *Walsh's* theorem.

If $w'=T(w)$ is a given one-valued, schlicht analytic function of w, we say that a region D of the extended w-plane has property T provided that the point $w'=T(w)$ lies in D, whenever w lies in D.

In [1], *J. L. Walsh extended the results of Radó, Seidel and Ford* [2] and proved the following theorem.

Suppose that a domain D has property T, where $T(w)$ is a linear transformation of the form;

$$(1) \qquad T: \frac{w'}{w'-a} = A \frac{w}{w-a}, \quad (a \neq 0,\ A \neq 1,\ 0 < |A| \leqq 1).$$

If the origin $w=0$ lies in D and Green's function $G(w)$ for D with the pole at the point $w=0$ exists, then every subregion D_γ ; $\{w\,;\, G(w) > -\log\gamma,\ 0<\gamma<1\}$ has also property T.

Futher he proved the following : *Let $T(w)$ be a one-valued, analytic function in D with $T(0)=0$ and suppose that a region D has property T and that there exists Green's function of D. Then it holds that*

$$(2) \qquad |T'(0)| \leqq 1.$$

In the case where $|T'(0)|=1$, or when one point and its image $T(w)$ simultaneously lies on the boundary of some D_γ, $T(w)$ becomes a schlicht function which maps D onto itself.

§**2.** Now we shall prove the following.

Theorem 1. *Suppose that a region D of the extended w-plane has property T, where $T(w)$ is a one-valued schlicht analytic function with $T(0)=0$. If $w=0$ lies in D and Green's function $G(w)$ for D with the pole at $w=0$ exists, then every subregion D_γ has also property T.*

Proof. By *Walsh's* method, our theorem is proved as follows. Let D' denote the image of D by $w'=T(w)$. By hypothesis $D'=T(D) \subseteqq D$. Since Green's function $G'(w')$ for D' with the pole at $w=0$ is the transform of $G(w)$, we have $G'(w') \equiv G[T^{-1}(w')]$.

If we consider the difference $u(w')=G(w')-G'(w')$, then $u(w')$ is harmonic in D' and $\geqq 0$ for every boundary point w' of D'. By the maximumprinciple, we have $G(w') \geqq G'(w')$ in D'. Consequently, whenever w lies in D, we have $G[T(w)]\geqq G(w)$. Hence, if w lies in D, that is, if $G(w)>-\log\gamma$, we have $G[T(w)]>-\log\gamma$ and hence $T(w)$ lies in D_γ. q. E. D.

If we restrict ourselves to the function $T(w)$ with the special condition in Theorem 1, we have the following theorem.

Theorem 2. *In Theorem* 1, *if* $T(w)$ *is a function which is transformed into a linear transformation* $S(z)$ *by a suitably defined analytic function* $z=f(w)$, *that is,*

(3) $$S(z)=fTf^{-1}(z)$$

becomes a linear transformation, $S(z)$ *is of the form*

(4) $$S:\frac{z'}{z'-a}=A\frac{z}{z-a}, \qquad (a\neq 0,\ A\neq 1,\ 0<|A|\leqq 1);$$

the case $a=\infty$ *being to be interpreted as* $z'=Az$.

If we take the identical transformation instead of $f(w)$ in (3), the above theorem becomes the *Walsh's* theorem in itself.

Proof. From (2), we have

$$|S'(0)|=|f'(0)|\ |T'(0)|\left|\frac{1}{f'(0)}\right|\leqq 1.$$

Since $S'(0)$ denotes the multiplier of a linear transformation, it is sufficient to prove the theorem in the case $|T'(0)|=1$ where $T(w)$ is a one-to-one conformal mapping of D onto itself. Because in the other case $T(w)$ becomes the linear transformation which is obtained from the linear transformation corresponding to the one-to-one conformal mapping $T(w)$ of D onto itself by multiplying a multiplier.

The trivial case where $T(w)$ is the identity w is excluded.

A region D_γ bounded by the level curve of Green's function has meaning only if the pole $w=0$ of Green's function lies in D.

Since every D_γ has property T with $T(0)=0$, we find at first $w=0$ must be a fixed point of $T(w)$.

If a region D has property T, it has also properties T^2, T^3, …….

Since, now, $T(w)$ is a one-to-one conformal mapping of D onto itself, then the functions

$$f(w)=T^n(w), \qquad (n=0,\ 1,\ 2,\dots),$$

which are generated from $T(w)$ and the inverse $T^{-1}(w)$ by the iteration, are also one-to-one conformal mappings of D onto itself. Hence the set $\{f_n(w)\}$ of all those transformations form a finite or an infinite cyclic group G.

Now we divide G into two cases.

i) If G is an infinite cyclic group and moreover contains no infinitesimal transformations, that is, G is the properly discontinuos group, there exist two or one *singular points*[3], which remain invariant under $T(w)\in G$, in the boundaries of D.

If there exist two *singular points*, one of them always coincides with $w=0$, and the other can be transformed into infinity without loss of generality. Since $w=0$ is an isolated boundary point from the property of D, it is removable from the hypothesis of $T(w)$. Then a neighborhood of $w=0$ contains a point which is congruent to any point of the extended plane and the region, which remains invariant under $T(w) \varepsilon G$, coincides with the extended plane. This consequence is obviously inconsistent with the property of D.

Even if there exists only one singular point, from the reason analogous to the above we can lead to the contradiction.

Therefore G has no singular points and is a finite group.

ii) If G is an infinite cyclic group and contain the infinitesimal transformations, G is a continuous group and there exist infinite *singular points*. We note that $w=0$ is a fixed inner point of D, hence it is not a *singular point*.

After all G is a finite or a special infinite group.

When the universal covering surface D^∞ of D is mapped onto the unit circle $|z|<1$ by the polymorphic function $z=f(w)$, so that $w=O$ corresponds to the origin of $|z|<1$, every element of G corresponds to a linear transformation SL, which holds $|z|<1$ invariant while L moves over the Fuchsian group Γ_0 of the automorphic function $w=w(z)$ $=f^{-1}(w)$. All such transformations SL form a Fuchsian group Γ, in which Γ_0 is contained as the invariant subgroup. Therefore we have the relation $G \cong \Gamma/\Gamma_0$, that is, G is isomorphic with the factor group Γ/Γ_0 which is a elliptic cyclic group[4].

Thus we obtain the relation $S=fTf^{-1}$ and $S(z)$ has the form (4) in Theorem 2. This is the consequence to be proved. q. E. D.

§ **3.** On the linearlization of the one-to-one conformal mapping
$T(w)$ of D onto itself.

The subject of this paragraph means that, if we can decide a suitably defined analytic function in D, $T(w)$ can be made linear by it. In other words, it is the problem to decide a conformal mapping function $z=f(w)$ from D onto a schlicht domain D' of the extended z-plane, so that the transformed function $s=fTf^{-1}$ may be linear, and of the form

(5) $$z'=S(z)=Az \qquad (|A|=1).$$

Let m be a closed set of the boundary points of D. We can suppose that the set m contains the points which belong only to the *transfinite kernel*,[5] for the other point, for instance, the isolated point, is removable. Generally, the boundary consists of continums and the points of the discrete set.

In the case, when G is an infinite group, the problem is difficult. But it is not difficult in § 2. Without loss of generality, we can place two fixed points at $w=0$ and infinity. We draw a Jordan curve k and the images k_ν of k obtained by T^ν, that is,

(6) $$k_\nu = T^\nu(k), \qquad (\nu=1,2,\dots, n-1).$$

If we denote the angular region bounded by k_ν and $k_{\nu+1}$ with $R_\nu(k_\nu, k_{\nu+1})$, so $\sum_{\nu=0}^{n-1} R_\nu$ cover the extended w-plane without overlappings and gaps. Suppose that m_0 is the subset of

m in R_o and $m=\sum_{\nu=0}^{n-1} T^{\nu}(m_o)$. If m_o is surrounded by a finite number of closed Jordan curves

(7) $C^{(o)}_{\mu,n}$, $(\mu=1,2,\ldots,n)$

in $D\cap R_o$, m_ν is also surrounded by the images

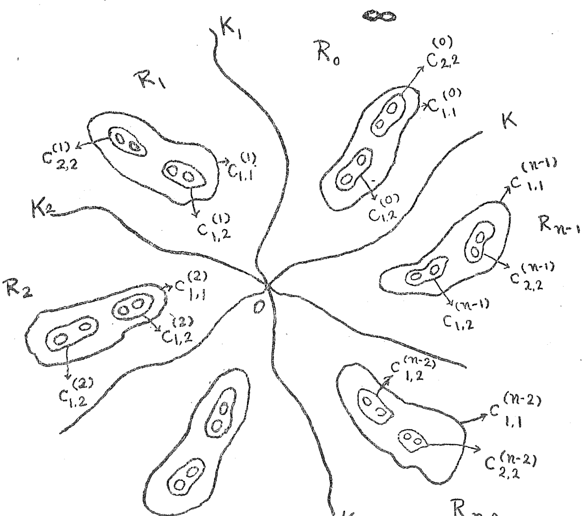
(8) $C^{(\nu)}_{\mu,n}$, $(\mu=1,2,\ldots,n)$

of (7) in $D\cap R_\nu$. (c. f. Fig.)

By the conformal mapping

(9) $z_n=f_n(w)$,

a subregion D_n, which are formed from D' by removing (7), (8) and their inside, can be mapped conformally onto a region B_n of the z-plane bounded by a circle with radius γ_n. Therefore $T(w)$ is transformed into $S_n(z)$ by $f_n(w)$. As $S_n(z)$ is a schlicht conformal mapping of B_n onto itself and is linear, so it is transformed into the form



Fig

(10) $z'_n=A_n z_n$, $(|A_n|=1)$.

Next we consider the infinite sequence of the system of curves

(11) $C^{(o)}_{\mu,n}$, $(\mu=1,2,3,\cdots n)$ $(n=1,2,3,\ldots.)$

which satisfy the following conditions;

1). The curves of the system $C^{(o)}_{\mu,n+1}$ is inside of the curves of the system $C^{(o)}_{\mu,n}$.

2). Any point w of D belongs to D_n for sufficiently large n.

From the infinite sequence $\{D_n\}$, we obtain $D=\lim_{n\to\infty} D_n$.

Now we normalize the infinite sequence $\{f_n(z)\}$, so that in any given point w_o of D $|f'_n(w_o)|=1$ may establish for all n. The functions $f_n(w)$, which map schlicht conformally any given compact subregion of D for sufficiently large n, are bounded by *Koebe's* distortion theorem. Thus $\{f_n(z)\}$ form a normal family and hence for a suitablely chosen subsequence there exists a non-constant, schlicht analytic limit function in D

(12) $\lim_{n\to\infty} f_{\nu_n}(w)=f(w)$.

Therefore $T(w)$ is transformed into a linear transformation

(13) $Z'=AZ$ $(|A|=1)$

by $f(w)$, where $A=\lim_{n\to\infty} A_{\nu_n}$. Thus our theorem is completely proved.

§ 4. On the existence of the non-linear one-to-one conformal mapping of a domain D onto itself.

If the boundaries of D consist of continums only, the existence of the above transformation is obvious. In the case where the boundaries are formed entirely by a discrete point set, the existence of such a transformation can be proved in the following way.

We begin an elliptic transformation

$$S: \qquad z' = Az, \qquad (|A| = 1)$$

and we take a discrete point set m of 2-dimensional positive outer measure in the angular region R_0 :

$$(14) \qquad \frac{2\pi\nu}{n} \leqq \arg z < \frac{2\pi(\nu+1)}{n}, \quad (\nu=0,1,2,\ldots n-1)$$

Then the 2-dimensional outer measure of the image

$$(15) \qquad m_\nu = S^\nu(m) \qquad (\nu=1,2,\ldots, n-1)$$

is also positive.

Now let $w=f(z)$ be a regular analytic function in the extended z-plane without the set m deleted onto a schlicht region D', the boundaries of which are a discrete point set m' of 2-dimensional measure zero.

Then the transformed function $T=f\ S\ f^{-1}$ is obviously nonlinear. Because by such a function the set m' corresponds with the set $f(m)$ of 2-dimensional positive outer measure.

References

[1] J. L. Walsh, *Note on the shape of level curves of Green's function*, *Duke Math. Jour.*, Vol. **20** (1953) pp 611-615.

[2] L. R. Ford, *On the properties of regions which persists in the subregions bounded by the level curves of the Green's function*, *Duke Math. Jour.*, vol. **1** (1935) pp 103-104.

[3] We call a point z of the plane *a regular point* of the group G, if there exists a neighborhood U(z) of z, so that there may exist only a finite number of images of U(z), which have common points with U(z), under the images of U(z) transformed by G. All other points of the plane are called *singluar points*.

[4] T. Akaza, *On the subregions bounded by the level curves of the Green's function*, *Sci. Rep. Kanazawa Univ*. Vo. **3** No. 1. (1955). pp 1—3.

[5] P. J. Myrberg, *Über die Existenz der Greenschen Funktionen auf einer gegebenen Riemannschen Fläche*, *Acta Math*. vol. **61** (1933) pp 39—79.