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On Some Property of a Gap Sequence

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1. Let $f(t)$ be a (B) measurable function with period 1. Suppose that for some constant $M>0$

(1.1) $$|f(t)|\leqq M$$

and

(1.2) $$\int_0^1 f(t)dt=0.$$

R. Fortet [1] and G. Maruyama [2] proved that if $f(t)$ satisfies the Lipschitz condition of order α $(0<\alpha\leqq 1)$, then

(1.3) $$\overline{\lim_{n\to\infty}}\frac{\sum_{k=1}^{n} f(2^k t)}{\sqrt{2n\log_2 n}}=\sigma$$

holds for almost all t, where

(1.4) $$\sigma^2=\lim_{n\to\infty}\frac{1}{n}\int_0^1\left(\sum_{k=1}^{n} f(2^k t)\right)^2 dt.$$

And also S. Izumi [3] obtained the same result when $f(t)$ is the function of Lip* $(1, \alpha)$ for $\alpha>0$.

The purpose of this paper is to sharpen the preceeding results. In §§ 2–4, we prove the following

THEOREM. Let $\{\varphi_k\}$ $k=1, 2, \cdots\cdots$ be an increasing sequence of positive numbers and $f(t)$ be the bounded function of Lip* $(1, \alpha)$ satisyfing (1.2), then for almost all t, there exist infinitely (or only finitely) many n such that

(1.5) $$\sum_{k=1}^{n} f(2^k t)>\sigma\sqrt{n}\,\varphi_n$$

if, and only if, the series

(1.6) $$\sum\frac{\varphi_k}{k}e^{-\frac{1}{2}\varphi_k^2}$$

diverges (or converges).

The proof is based on the following theorem of W. Feller [4] .Let{ x_k}$k=1, 2, \cdots\cdots$

Lip* $(1, \alpha)$ means that $\int_0^1 \max_{h\leqq u}|f(x+h)-f(x)|dx=O(u^\alpha)$.

be a sequence of independent random variables with mean value zero. Suppose that

$$E(x_k^2)=\sigma_k^2$$

exists and

$$s_n^2=\sum_{k=1}^{n}\sigma_k^2\to+\infty \qquad (n\to+\infty),$$

then the theorem of W. Feller reads as follows.

THEOREM A. If

$$|x_k|\leqq M_k=O\left(\frac{s_k}{\varphi_k^{3/2}}\right) \qquad (k\to+\infty),$$

then with probability one, there exist infinitely (or only finitely) many n such that

$$\sum_{k=1}^{n}x_k>s_n\left(\varphi_n+\frac{C}{\varphi_n}\right)$$

if, and only if, the series

$$\sum\frac{\sigma_k^2}{s_k^2}\varphi_k e^{-\frac{1}{2}\varphi_k^2}$$

diverges (or converges), where C is any constant.

2. We use the following

LEMMA 1. Without loss of generality, we may assume that

$$\log_2 k\leqq\varphi_k^2\leqq 4\log_2 k.$$

The proof is the same as that of lemma 1 in [5] .

LEMMA 2. We have

$$\rho_n\equiv n\sigma^2-\int_0^1\left(\sum_{k=1}^{n}f(2^k t)\right)^2 dt=O(1) \qquad (n\to+\infty).$$

PROOF. We have, after M. Kac [6]

$$\frac{1}{n}\int_0^1\left(\sum_{k=1}^{n}f(2^k t)\right)^2 dt$$

$$=\int_0^1 f^2(t)dt+2\sum_{k=1}^{n}\int_0^1 f(t)f(2^k t)dt-\frac{2}{n}\sum_{k=1}^{n}k\int_0^1 f(t)f(2^k t)dt.$$

By lemma 1 of S. Izumi [7], it is known that

$$\left|\int_0^1 f(t)f(2^k t)dt\right|\leqq\frac{A}{2^{\alpha k}}.$$

Hence the limit (1.4) exists and we have

$$|\rho_n|=n\left|\lim_{m\to\infty}\frac{1}{m}\int_0^1\left(\sum_{k=1}^{m}f(2^k t)\right)^2 dt-\frac{1}{n}\int_0^1\left(\sum_{k=1}^{n}f(2^k t)\right)^2 dt\right|$$

$$=n\lim_{m\to\infty}\left|\sum_{k=n+1}^{m}\int_0^1 f(t)f(2^k t)dt\right|+O(1)=O(1).$$

The following definitions of sequences are essentially the same as those of Maruyama in [2]. Let us put

$$\lambda(\nu)=[2\log\nu/a\log 2] \tag{2.1}$$

and denote N_0 the smallest integer k such as

$$[(\log k)^2]>\lambda(k(\log k)^2)\ . \tag{2.2}$$

For $k\geqq N_0$, let us define sequences $\{m_k\}$ and $\{n_k\}$ as follows.

$$m_k=[k(\log k)^2]\ , \tag{2.3}$$

and

$$n_k=m_k+\lambda(m_k)\ . \tag{2.4}$$

From (2.2), (2.3) and (2.4), it is easily seen that

$$n_{k-1}<m_k<n_k\ ,$$

$$l_k=m_k-n_{k-1}\sim(\log k)^2 \qquad (k\to\infty), \tag{2.5}$$

and

$$\lambda(m_k)=n_k-m_k\sim\frac{2\log k}{a\log 2} \qquad (k\to+\infty). \tag{2.6}$$

LEMMA 3. The series (1.6) and

$$\sum\frac{\varphi_{m_k}}{k}\ e^{-\frac{1}{2}\varphi^2_{m_k}} \tag{2.7}$$

converges or diverges simultaneously.

PROOF. Since $\{\varphi_k\}$ is increasing, we have

$$\varphi_{m_k}\,e^{-\frac{1}{2}\varphi^2_{m_k}}\leqq\varphi_k e^{-\frac{1}{2}\varphi^2_k}\ ,$$

hence the convergence of (1.6) implies that of (2.7).

Conversely, if (2.7) converges, then

$$\sum\frac{\varphi_k}{k}e^{-\frac{1}{2}\varphi^2_k}=\sum_k\sum_{\nu=m_k}^{m_{k+1}-1}\frac{\varphi_\nu}{\nu}e^{-\frac{1}{2}\varphi^2_\nu}$$

$$\leqq\sum_k\varphi_{m_k}\,e^{-\frac{1}{2}\varphi^2_{m_k}}\log\frac{m_{k+1}}{m_k}\ .$$

By the definition of $\{m_k\}$, we have for $k\geqq$Max $[3,\ N_0]$

$$\log\frac{m_{k+1}}{m_k}\leqq\log\frac{(k+1)(\log(k+1))^2}{(k-1)(\log k)^2}$$

$$\leqq\log\left(1+\frac{2}{k-1}\right)+2\log\left(1+\frac{\log\left(1+\frac{1}{k}\right)}{\log k}\right)$$

$$\leqq\frac{2}{k-1}+\frac{2}{k\log k}\leqq\frac{5}{k}\ .$$

This shows that (1.6) converges.

3. Let t be a point in (0.1) and its dyadic expansion be

$$t=\sum_{k=1}^{\infty}\frac{d_k(t)}{2^k}\ ,$$

where $d_k(t)=0$ or 1. Using Rademacher functions $r_k(t)$, we may write

$$d_k(t)=\frac{1+r_k(t)}{2} \qquad k=1,\ 2,\cdots\cdots. \tag{3.1}$$

If we put

$$\theta_\nu(t)=\sum_{k=1}^{\lambda(\nu)}\frac{d_{\nu+k}(t)}{2^k} \tag{3.2}$$

then it follows that

$$2^\nu t-\theta_\nu(t)=\sum_{k=1}^{\nu}2^{\nu-k}d_k+O\left(\frac{1}{2^{\lambda(\nu)}}\right) \qquad (\nu\to+\infty), \tag{3.3}$$

and

$$\begin{aligned} f(2^\nu t)&=\{f(2^\nu t)-f(\theta_\nu(t))\}+f(\theta_\nu(t)) \\ &\equiv p_\nu(t)+f(\theta_\nu(t)), \end{aligned} \tag{3.4}$$

whence by the hypothesis and (3.3), we have

$$\int_0^1|p_\nu(t)|\,dt\leqq\frac{A}{2^{\alpha\lambda(\nu)}}=\frac{A}{\nu^2},$$

where A is a constant independent of ν.
This implies that

$$\sum_\nu|p_\nu(t)|<+\infty \qquad \text{a. e. .} \tag{3.5}$$

Let us define

$$x_k(t)=\sum_{\nu=n_{k-1}+1}^{m_k}\{f(\theta_\nu(t))-e_\nu\} \qquad \text{for } k>N_0, \tag{3.6}$$

and

$$y_k(t)=\sum_{\nu=m_k+1}^{n_k}\{f(\theta_\nu(t))-e_\nu\} \qquad \text{for } k\geqq N_0, \tag{3.7}$$

where

$$e_\nu=\int_0^1 f(\theta_\nu(t))dt.$$

It is easily seen

$$\begin{aligned} |e_\nu|&\leqq\int_0^1|f(\theta_\nu(t))-f(2^\nu t)|\,dt+\left|\int_0^1 f(2^\nu t)dt\right| \\ &\leqq\frac{A}{\nu^2}. \end{aligned} \tag{3.8}$$

Since $x_k(t)$ is the (B) measurable function of the Rademacher functions $r_n(t)$ where

$$n_{k-1}<n\leqq m_k+\lambda(m_k)=n_k,$$

$x_k(t)$ and $x_{k+1}(t)$ do not contain the same Rademacher functions, and $\{x_k(t)\}$ is the sequence of independent functions having mean value zero. By the same way, we see that $\{y_k(t)\}$ is the sequence of independent functions with mean value zero.

On the other hand we have

$$\sigma_k^2 \equiv \int_0^1 (x_k(t))^2 dt$$

$$= \int_0^1 \Big\{ \sum_{\nu = n_{k-1}+1}^{m_k} (f(2^\nu t) + f(\theta_\nu(t)) - f(2^\nu t) - e_\nu) \Big\}^2 dt$$

$$= \int_0^1 \Big(\sum_{\nu = n_{k-1}+1}^{m_k} f(2^\nu t) \Big)^2 dt + \int_0^1 \Big\{ \sum_{\nu = n_{k-1}+1}^{m_k} (f(\theta_\nu(t)) - f(2^\nu t) - e_\nu) \Big\}^2 dt$$

$$+ 2 \int_0^1 \Big(\sum_{\nu = n_{k-1}+1}^{m_k} f(2^\nu t) \Big) \Big\{ \sum_{\nu = n_{k-1}+1}^{m_k} (f(\theta_\nu(t)) - f(2^\nu t) - e_\nu) \Big\} dt$$

$$\equiv P_k + Q_k + 2R_k.$$

Then by Lemma 2, we have

$$P_k = \sigma^2 (m_k - n_{k-1}) + O(1) = O((\log k)^2) \qquad (k \to +\infty).$$

And by (1.1) and (3.8), we have

$$I_{\mu,\nu} \equiv \Big| \int_0^1 [f(\theta_\nu(t)) - f(2^\nu t) - e_\nu][f(\theta_\mu(t)) - f(2^\mu t) - e_\mu] dt \Big|$$

$$\leqq M \Big\{ \int_0^1 |(f(\theta_\nu(t)) - f(2^\nu t)| dt + |e_\nu| \Big\}$$

$$= O\Big(\frac{1}{\nu^2}\Big) = O\Big(\frac{1}{n_{k-1}^2}\Big) = O\Big(\frac{1}{k^2 (\log k)^4}\Big) \qquad (k \to +\infty),$$

hence

$$Q_k \leqq (m_k - n_{k-1})^2 \operatorname*{Max}_{n_{k-1} < \mu, \nu \leqq m_k} I_{\mu,\nu} = O\Big(\frac{1}{k^2}\Big) \qquad (k \to +\infty).$$

By the above results and Schwarz inequality we have

$$|R_k| \leqq P_k^{1/2} \quad Q_k^{1/2} = O\Big(\frac{\log k}{k}\Big) \qquad (k \to +\infty).$$

Therefore, we obtain

(3.9)
$$\sigma_k^2 = (m_k - n_{k-1}) \sigma^2 + O(1) \sim \sigma^2 (\log k)^2 \qquad (k \to +\infty),$$

and by the same way

(3.10)
$$\tau_k^2 \equiv \int_0^1 (y_k(t))^2 dt = \lambda(m_k) \sigma^2 + O(1) \qquad (k \to +\infty).$$

Now we have

(3.11)
$$B_N^2 \equiv \sum_{k = N_0+1}^{N} \sigma_k^2$$

$$= \sigma^2 \sum_{k = N_0+1}^{N} [(m_k - m_{k-1}) + (m_{k-1} - n_{k-1})] + O(N)$$

$$= \sigma^2 m_N + O(N \log N) \sim \sigma^2 N (\log N)^2 \qquad (N \to +\infty).$$

Hence, we have, by (2.3) and Lemma 1,

(3.12)
$$B_N = \sigma \sqrt{m_N} + O(\sqrt{N}) = \sigma \sqrt{m_N} + o\Big(\frac{\sqrt{m_N}}{\varphi_{m_N}^2}\Big) \qquad (N \to +\infty).$$

On the other hand

$$C_N^2 \equiv \sum_{k=N_0}^{N} \tau_k^2 = \sigma^2 \sum_{k=N_0}^{N} \lambda(m_k) + O(N) \sim \frac{\sigma^2 2N \log N}{\alpha \log 2} \qquad (N \to +\infty). \tag{3.13}$$

Therefore by (1.1), (3.7) and (3.13), we have

$$|y_k(t)| < 2\lambda(m_k) M = o\left(\frac{C_k}{\log_2{}^{1/2} C_k}\right) \qquad (k \to +\infty),$$

and if we apply A. Kolmogoroff's theorem of the iterated logarithm, it follows that for almost all t,

$$\overline{\lim_{N \to +\infty}} \frac{\left|\sum_{k=N_0}^{N} y_k(t)\right|}{(2C_N^2 \log_2 C_N)^{1/2}} = 1. \tag{3.14}$$

From (3.13) and (3.14), we have

$$\lim_{N \to \infty} \frac{\sum_{k=N_0}^{N} y_k(t)}{\sqrt{\dfrac{m_N}{\varphi_{m_N}}}} = 0 \qquad \text{a. e..} \tag{3.15}$$

On the other hand, by the definitions of $x_k(t)$, $y_k(t)$ and $p_\nu(t)$, we have

$$\sum_{k=1}^{m_N} f(2^k t) = \sum_{k=1}^{m_{N_0}} f(2^k t) + \sum_{k=N_0}^{N-1} y_k(t) + \sum_{k=N_0+1}^{N} x_k(t)$$

$$+ \sum_{\nu=m_{N_0}+1}^{m_N} p_\nu + \sum_{\nu=m_{N_0}+1}^{m_N} e_\nu ,$$

and by (1.1), (3.5), (3.8) and (3.15) we have

$$\sum_{k=1}^{m_N} f(2^k t) = \sum_{k=N_0+1}^{N} x_k(t) + o\left(\frac{\sqrt{m_N}}{\varphi_{m_N}}\right) \qquad \text{a. e..} \tag{3.16}$$

4. Now from (3.9) and (3.11), we can find two positive constants A_1 and A_2 such as

$$\frac{A_1}{k} \leqq \frac{\sigma_k^2}{B_k^2} \leqq \frac{A_2}{k} \qquad \text{for } k > N_0.$$

Hence by Lemma 3, the two series (1.6) and

$$\sum \frac{\sigma_k^2}{B_k^2} \varphi_{m_k} e^{-\frac{1}{2}\varphi_{m_k}^2} \tag{4.1}$$

converges or diverges simultaneously.

On the other hand by (1.1), (3.6) and (3.12) we have

$$|x_k(t)| \leqq 2M l_k = O\left(\frac{B_k}{\varphi_{m_k}^{3/2}}\right) \qquad (k \to +\infty). \tag{4.2}$$

Now, Theorem A in §1 is applicable for $\{x_k(t)\}$.

1°. The case where (1.6) diverges.

In this case, (4.1) diverges, and we have, for almost all t

$$\sum_{k=N_0+1}^{N} x_k(t) > B_N\left(\varphi_{m_N} + \frac{C}{\varphi_{m_N}}\right) \tag{4.3}$$

occurs for infinitely many N.

Hence, for almost all t,

$$\sum_{k=1}^{m_{N}} f(2^k t) > B_N\left(\varphi_{m_N} + \frac{C}{\varphi_{m_N}}\right) + o\left(\frac{\sqrt{m_N}}{\varphi_{m_N}}\right)$$

$$= \sigma\sqrt{m_N}\,\varphi_{m_N} + C\sigma\frac{\sqrt{m_N}}{\varphi_{m_N}} + o\left(\frac{\sqrt{m_N}}{\varphi_{m_N}}\right) \qquad (N\to+\infty),$$

occurs for infinitely many N, by (3.15) and (3.12).

If we put $C=1$, then for almost all t and sufficiently large N

$$\sum_{k=1}^{m_N} f(2^k t) > \sigma\sqrt{m_N}\,\varphi_{m_N},$$

therefore, we have infinitely many n such that

$$\sum_{k=1}^{n} f(2^k t) > \sigma\sqrt{n}\,\varphi_n \qquad \text{a. e..}$$

2°. The case where (1.6) converges.

In this case, (4.1) converges and we have, for almost all t,

$$\text{(4.5)} \qquad \sum_{k=N_0+1}^{N} x_k(t) > B_N\left(\varphi_{m_N} + \frac{C'}{\varphi_{m_N}}\right)$$

occurs only finitely many N.

For any n, we can find $N(n)$ such that

$$m_{N(n)} < n < m_{N(n)+1}$$

and by (1.1), we have for all t

$$\text{(4.6)} \qquad \sum_{k=m_{N(n)}+1}^{n} f(2^k t) \leqq (m_{N(n)+1} - m_{N(n)})M$$

$$= O\left((\log N(n))^2\right) = o\left(\frac{\sqrt{m_{N(n)}}}{\varphi_{m_{N(n)}}}\right) \qquad (n\to+\infty).$$

From (4.6) and (3.15), we have for almost all t

$$\text{(4.7)} \qquad \sum_{k=1}^{n} f(2^k t) = \sum_{k=N_0+1}^{N(n)} x_k(t) + o\left(\frac{\sqrt{m_{N(n)}}}{\varphi_{m_{N(n)}}}\right) \qquad (n\to+\infty).$$

Excluding a set of measure zero, we can find $K(t_0)$ for any fixed t_0 such that $N > K(t_0)$ implies

$$\sum_{k=N_0+1}^{N} x_k(t_0) \leqq B_N\left(\varphi_{m_N} + \frac{C'}{\varphi_{m_N}}\right).$$

Hence by (4.7), if $N(n) > K(t_0)$, we have

$$\sum_{1}^{n} f(2^k t_0) = \sum_{k=N_0+1}^{N(n)} x_k(t_0) + o\left(\frac{\sqrt{m_{N(n)}}}{\varphi_{m_{N(n)}}}\right)$$

$$\leqq B_{N(n)}\left(\varphi_{m_{N(n)}} + \frac{C'}{\varphi_{m_{N(n)}}}\right) + o\left(\frac{\sqrt{m_{N(n)}}}{\varphi_{m_{N(n)}}}\right)$$

$$= \sigma\sqrt{m_{N(n)}}\,\varphi_{m_{N(n)}} + C'\sigma\frac{\sqrt{m_{N(n)}}}{\varphi_{m_{N(n)}}} + o\left(\frac{\sqrt{m_{N(n)}}}{\varphi_{m_{N(n)}}}\right).$$

Consequently if we put $C'=-1$, we can find $K'(t_0)$ such that $N(n)>[K(t_0)$ and $K'(t_0)]$ implies

$$\sigma\sqrt{m_{N(n)}}\,\varphi_{m_{N(n)}}+\sigma C'\frac{\sqrt{m_{N(n)}}}{\varphi_{m_{N(n)}}}+o\left(\frac{\sqrt{m_{N(n)}}}{\varphi_{m_{N(n)}}}\right)$$
$$\leqq\sigma\sqrt{m_{N(n)}}\varphi_{m_{N(n)}}<\sqrt{n}\,\sigma\varphi_n.$$

Hence excluding a set of measure zero, we can find $K''(t_0)$ for any fixed t_0 such that $n>K''(t_0)$ implies

$$\sum_{k=1}^{n} f(2^k t_0)<\sigma\sqrt{n}\,\varphi_n.$$

Thus the theorem is proved.

References

[1] R. Fortet, Sur une suite egalement repartie, Studia Math., vol. 9 (1940) pp. 54—69.

[2] G. Maruyama, On asymptotic property of a gap sequences, Kodai Seminary Report, vol. (1950) pp. 31—32.

[3] S. Izumi, Notes on Fourier Analysis (XLIV), Jour. of Math., vol. 1 (1952) pp. 1—22.

[4] W. Feller, General form of the so called law of the iterated logarithm, Trans. Amer. Math. Soc., vol. 54 (1943) pp. 372—402. (Theorem 2).

[5] W. Feller, The law of the iterated logarithm for identically distributed random variables, Ann. of Math., vol. 47 (1946) pp. 631—638.

[6] M. Kac, On the distributions of sums of the type $\frac{1}{\sqrt{n}}\sum_{1}^{n} f(2^k t)$, Ann. of Math., vol. 47 (1946) pp. 33—49.

[7] S. Izumi, Notes on Fourier Analysis (XVI). Tohoku Math. Jour., vol. 1 (1950) pp. 144—166 (Part IV. Lemma 1).