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On the Law of the Iterated Logarithm

By

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1. Let $\{Z_k\}$ $k=1, 2, \cdots\cdots$ be an infinite sequence of independent random variables and $V_k(x)$ be the distribution function of Z_k. Suppose that the second moment

(1.1) $$b_k^2=\int_{-\infty}^{+\infty} x^2 dV_k(x)$$

exists and the mean value

(1.2) $$\int_{-\infty}^{+\infty} x dV_k(x)=0,$$

for $k=1,2,\cdots\cdots$.

Let B_n denote the dispersion of $\sum_{k=1}^{n} Z_k$, i.e.,

(1.3) $$B_n=\sum_{k=1}^{n} b_k^2.$$

A. Kolmogoroff [1] proved the following which is called the Kolmogoroff's theorem of the law of iterated logarithm.

THEOREM α. If

(1.4) $$B_n \uparrow +\infty$$

as $n\to+\infty$ and

(1.5) $$|Z_n|\leq m_n=o\left(\frac{B_n}{\log_2 B_n}\right)^{1/2}$$

as $n\to+\infty$, then

$$\mathrm{Prob}\left[\overline{\lim_{n\to\infty}}\frac{Z_1+Z_2+\cdots\cdots+Z_n}{(2B_n\log_2 B_n)^{1/2}}=1\right]=1.$$

On the other hand, Hartman and Wintner [2] proved the following

THEOREM β. Suppose that

(1.6) $$B_n>n\,\mathrm{const}>0$$

and that the $V_k(x)$ possess a dominant in the following sense:

There exists a distribution function $V(x)$, which has the second moment and is such that

$$\int_{|x|\geq r} dV_k(x)=O\left(\int_{|x|\geq r} dV(x)\right)$$

$r\to+\infty$ holds uniformly in k. Then

$$\mathrm{Prob}\left[\overline{\lim_{n\to\infty}}\frac{Z_1+\cdots\cdots+Z_n}{(2B_n\log_2 B_n)^{1/2}}=1\right]=1.$$

From the conditions of this theorem, it is seen that

(1.7) $$b_n^2 = O(1)$$

as $n \to +\infty$, and there exist two constants $0 < \alpha \leqq \beta$ such that

(1.8) $$n\alpha \leqq B_n \leqq n\beta.$$

The purpose of this paper is to replace the condition (1.5) by some conditions concerning dispersions (c. f. (1.8)).

Let $\{X_k\}$ $k=1, 2, \cdots\cdots$ be a sequence of independent random variables and $F_k(x)$ be the distribution function of X_k. Suppose that

(1.9) $$\int_{-\infty}^{+\infty} x^2 dF_k(x) = 1,$$

and

(1.10) $$\int_{-\infty}^{+\infty} x dF_k(x) = 0, \qquad \text{for } k=1, 2, \cdots\cdots$$

Moreover let $\{b_k\}$ $k=1, 2, \cdots\cdots$ be any sequence of real numbers such that*)

(1.11) $$B_n = \sum_{k=1}^{n} b_k^2 \uparrow +\infty \qquad \text{as } n \to +\infty.$$

Under these conditions, we prove the following

THEOREM. Suppose that

(A) $$F_k(x) = F(x) \qquad \text{for } k=1, 2, \cdots\cdots$$

or

(B) $F_k(x)$ has a dominant $F(x)$ in the sense of Hartman and Wintner in the preceeding theorem.

If

(1.12) $$|b_n| = o(B_n^{1/2}) \qquad \text{as } n \to +\infty,$$

and

(1.13) $$|b_n| = O\left(\frac{B_n \log_2 B_n}{n \log_3 n}\right)^{1/2} \qquad \text{as } n \to +\infty,$$

then

$$\text{Prob}\left[\varlimsup_{n\to\infty} \frac{b_1 X_1 + b_2 X_2 + \cdots\cdots + b_n X_n}{(2B_n \log_2 B_n)^{1/2}} = 1\right] = 1.$$

In the case (B), it is seen that

$$\int_{|x|\geqq r} dF_k(x) = O\left(\int_{|x|\geqq r} dF(x)\right),$$

$$\int_{|x|\geqq r} |x| dF_k(x) = O\left(\int_{x\geqq r} |x| dF(x)\right),$$

and

$$\int_{|x|\geqq r} x^2 dF_k(x) = O\left(\int_{|x|\geqq r} x^2 dF(x)\right),$$

*) If $\sum b_k^2$ converges, then Prob $[\sum b_k X_k$ converges$]=1$.

hold uniformly in k (as $r\to+\infty$). Using above three relations, we can prove the theorem by the similar way as that of the case (A). Whence, for simplicity, we prove the theorem, in §§2—4, only for the case (A).

2. A simple computation [3] shows that (c.f. (1.11), (1.12) and (1.13))

$$\log B_n \sim \sum_{k=1}^{n} \frac{b_k^2}{B_k} = O(\log_2 B_n \cdot \log n)$$

as $n\to+\infty$, hence we obtain

$$\log_2 B_n = O(\log_2 n), \qquad n\to+\infty. \tag{2.1}$$

Let k_0 denote the smallest integer n such that

$$\log_3 n>0 \text{ and } \log_2 B_n>0.$$

Put

$$\alpha_k = \begin{cases} k^{1/2} & \text{for } 1\leqq k<k_0, \\ \dfrac{k^{1/2}}{\log_2 k} & \text{for } k_0\leqq k. \end{cases} \tag{2.2}$$

Define a sequence $\{X_k'\}$ of independent random variables as follows.

$$X_k' = \begin{cases} -\alpha_k' & \text{if } |X_k|>\alpha_k, \\ X_k-\alpha_k' & \text{if } |X_k|\leqq\alpha_k, \end{cases} \tag{2.3}$$

where

$$\alpha_k' = \int_{|x|<\alpha_k} x\,dF(x) \qquad \text{for } k=1,\ 2,\cdots\cdots. \tag{2.4}$$

Then, from the definition of X_k' and (1.10), we have, for $k=1,\ 2,\cdots\cdots$,

$$E(b_k X_k')=0$$

and

$$E[(b_k X_k')^2] = b_k^2\Big[\int_{|x|\leqq\alpha_k} x^2 dF(x) - \Big(\int_{|x|>\alpha_k} x\,dF(x)\Big)^2\Big].$$

If we denote for $n=1,\ 2,\cdots\cdots$

$$B_n' = \sum_{k=1}^{n} E[(b_k X_k')^2],$$

then by (1.9), we have

$$0\leqq B_n - B_n' = \sum_{k=1}^{n} b_k^2\Big[\int_{|x|>\alpha_k} x^2 dF(x) + \Big(\int_{|x|>\alpha_k} x\,dF(x)\Big)^2\Big]$$

$$\leqq 2\sum_{k=1}^{n} b_k^2 \int_{|x|>\alpha_k} x^2 dF(x)$$

$$\leqq 2\sum_{k=1}^{N_0} b_k^2 + 2\Big(\sum_{k=N_0+1}^{n} b_k^2\Big)\int_{|x|>\alpha_{N_0+1}} x^2 dF(x).$$

Thus for any $\varepsilon>0$, there exists N_0' such as, for $n\geqq N_0'$,

$$0\leqq B_n - B_n' \leqq 2B_{N_0} + 2B_n\varepsilon,$$

and hence by (1.11), we obtain

(2.5) $$B_n \sim B_n' \qquad n \to +\infty.$$

From the definition of X_n', (2.1) and (1.16), we have

$$|b_n X_n'| \leqq |b_n|(|a_n| + |a_n'|) \leqq 2|b_n a_n| \leqq |2b_n \frac{n^{1/2}}{\log_2 n}| = o\Big(\frac{B_n}{\log_2 B_n}\Big)^{1/2}$$

as $n \to \infty$. Hence by (2.5), it follows that

$$|b_n X_n'| = o\Big(\frac{B_n'}{\log_2 B_n'}\Big)^{1/2}$$

as $n \to \infty$. Applying Kolmogoroff's theorem in §1 to $\{b_k X_k'\}$, we obtain

$$\mathrm{Prob}\Big[\overline{\lim_{n\to\infty}} \frac{b_1 X_1' + \cdots\cdots + b_n X_n'}{(2B_n' \log_2 B_n')^{1/2}} = 1\Big] = 1,$$

and by (2.5)

(2.6) $$\mathrm{Prob}\Big[\overline{\lim_{n\to\infty}} \frac{b_1 X_1' + \cdots\cdots + b_n X_n'}{(2B_n \log_2 B_n)^{1/2}} = 1\Big] = 1.$$

3. Let us define a sequence $\{X_k''\}$ of independent random variables as follows. If $1 \leqq k < k_0$, then $X_k'' = 0$ and if $k \geqq k_0$, then

(3.1) $$X_k'' = \begin{cases} -a_k'' & \text{if } |X_k| > k^{1/2} \text{ or } |X_k| \leqq a_k, \\ X_k - a_k'' & \text{if } a_k < |X_k| \leqq k^{1/2}, \end{cases}$$

where

(3.2) $$a_k'' = \int_{a_k < |x| \leqq k^{1/2}} x\,dF(x).$$

From the definition of X_k'', we have

$$E(b_k X_k'') = 0,$$

and by (1.9)

$$E[(b_k X_k'')^2] = b_k^2\Big[\int_{a_k < |x| \leqq k^{1/2}} x^2 dF(x) - \Big(\int_{a_k < |x| \leqq k^{1/2}} x\,dF(x)\Big)^2\Big]$$

$$\leqq b_k^2 \int_{a_k < |x| \leqq k^{1/2}} x^2 dF(x) \qquad \text{for } k \geqq k_0.$$

By the above relation and (1.13), we have

$$\sum_{k=k_0}^{\infty} \frac{E[(b_k X_k'')^2]}{(B_k \log_2 B_k)} = O\Big(\sum_{k=k_0}^{\infty} \frac{1}{k \log_3 k} \int_{a_k < |x| \leqq k^{1/2}} x^2 dF(x)\Big)$$

$$= O\Big(\sum_{k=k_0}^{\infty} \frac{1}{k \log_3 k} \sum_{m=\left[\frac{k^{1/2}}{\log_2 k}\right]}^{[k^{1/2}]} \int_{m < |x| \leqq m+1} x^2 dF(x)\Big)$$

$$= O\Big(\sum_{m=\left[\frac{k_0^{1/2}}{\log_2 k_0}\right]}^{\infty} \int_{m < |x| \leqq m+1} x^2 dF(x) \sum_{k=m^2}^{[m^2 \log_2^3 m]} \frac{1}{k \log_3 k}\Big) + O(1)$$

$$=O\Big(\int_{|x|>\left[\frac{k_0^{1/2}}{\log_2 k_0}\right]} x^2 dF(x)\Big)+O(1)=O(1).$$

Hence, by the Tchebyscheff's inequality, it follows that

$$\sum_{k=k_0}^{\infty}\frac{b_k X_k''}{(B_k \log_2 B_k)^{1/2}} \text{ converges in probability,}$$

and by P. Lévy's theorem [4], this is equivalent to

$$\text{Prob}\Big[\sum_{k=k_0}^{\infty}\frac{b_k X_k''}{(B_k \log_2 B_k)^{1/2}} \text{ converges}\Big]=1.$$

Therefore by Knopp's theorem, we obtain

$$\text{(3.3)} \qquad \text{Prob}\Big[\lim_{n\to\infty}\frac{b_1 X_1''+\cdots\cdots+b_n X_n''}{(2B_n \log_2 B_n)^{1/2}}=0\Big]=1.$$

4. We define a sequence $\{X_k'''\}$ of independent random variables such as

$$\text{(4.1)} \qquad X_k'''=\begin{cases} X_k & \text{if } |X_k|>k^{1/2}, \\ 0 & \text{if } |X_k|\leqq k^{1/2}, \end{cases} \qquad =1,\ 2,\cdots\cdots.$$

Then from the definition of X_k''' and (1.9), we have

$$\sum_{k=1}^{\infty}\text{Prob}[|b_k X_k'''|\neq 0]\leqq\sum_{k=1}^{\infty}\text{Prob}[|X_k|>k^{1/2}]$$

$$=\sum_{k=1}^{\infty}\int_{|x|>k^{1/2}} dF(x)=\sum_{k=1}^{\infty} k\int_{(k+1)^{1/2}>x>k^{1/2}} dF(x)\leqq\int_{|x|>1} x^2 dF(x).$$

Hence we have

$$\text{Prob}\Big[\sum_{k=1}^{\infty}|b_k X_k'''| \text{ converges}\Big]=1.$$

Therefore by (1.11), we obtain

$$\text{(4.2)} \qquad \text{Prob}\Big[\lim_{n\to\infty}\frac{b_1 X_1'''+\cdots\cdots+b_n X_n'''}{(2B_n \log_2 B_n)^{1/2}}=0\Big]=1.$$

From (2.6), (3.3) and (4.2), it is seen that

$$\text{Prob}\Big[\overline{\lim_{n\to\infty}}\frac{\sum_{k=1}^{n} b_k(X_k'+X_k''+X_k''')}{(2B_n \log_2 B_n)^{1/2}}=1\Big]=1.$$

Now from the difinitions of X_k', X_k'' and X_k''', we have

$$b_k X_k-b_k(X_k'+X_k''+X_k''')$$

$$=b_k\int_{|x|\leqq k^{1/2}} x dF(x)=-b_k\int_{|x|>k^{1/2}} x dF(x) \qquad \text{for } k=1,\ 2,\cdots\cdots.$$

And we have by (1.9) and (1.13),

$$\sum_{k=k_0}^{\infty}\frac{|b_k|}{(B_k \log_2 B_k)^{1/2}}\int_{|x|>k^{1/2}}|x|\,dF(x)$$

$$=O\left(\sum_{k=k_0}^{n}\frac{1}{(k\log_3 k)^{1/2}}\sum_{m=[k^{1/2}]}^{[n^{1/2}]}\int_{(m+1)\geq x>m}|x|\,dF(x)\right)+O\left(\sum_{k=k_0}^{n}\frac{1}{(k\log_3 k)^{1/2}}\int_{|x|>n^{1/2}}|x|\,dF(x)\right)$$

$$=O\left(\sum_{m=[k_0^{1/2}]}^{[n^{1/2}]} m\int_{(m+1)\geq|x|>m}|x|\,dF(x)\sum_{k=k_0}^{m^2}\frac{1}{(k\log_3 k)^{1/2}}\right)+O\left(n^{1/2}\int_{|x|>n^{1/2}}|x|\,dF(x)\right)$$

$$=O\left(\sum_{m=[k_0^{1/2}]}^{[n^{1/2}]} m\int_{(m+1)\geq|x|>m}|x|\,dF(x)\right)+O\left(\int_{|x|>n^{1/2}}x^2\,dF(x)\right)$$

$$=O\left(\int_{|x|>[k_0^{1/2}]}x^2 dF(x)\right)+O\left(\int_{|x|>n^{1/2}}x^2 dF(x)\right)$$

$$=O(1) \qquad \text{as } n\to+\infty.$$

Hence by Konopp's theorem we obtain

$$\frac{\sum_{k=1}^{n}|b_kX_k-b_k(X_k'+X_k''+X_k''')|}{(2B_n\log_2 B_n)^{1/2}}=o(1) \qquad \text{as } n\to+\infty.$$

Therefore we have

$$\text{Prob}\left[\overline{\lim_{n\to\infty}}\frac{b_1X_1+\cdots\cdots+b_nX_n}{(2B_n\log_2 B_n)^{1/2}}=1\right]=1.$$

Thus we prove the Theorem.

References

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[4] P. Lévy, Théorie de l'addition de variables aléatoires, Paris (1937) pp. 139—140.