ιe Science Reports of the Kanazawa University, Vol. III, No. 1, pp. 5—9, March, 1955.

A Note on the general Divisor Problem

By

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(Received October 2, 1954)

1. Prof. Hasse and Prof. Suetuna first considered the general divisor problem in an algebraic number fields as follows [5] :

Ω is an extension of the rational number field of degree l and K is an extension of Ω of degree m.

$$A_\varkappa(x)=\sum_{N(\mathfrak{n})\leq x} T_\varkappa(\mathfrak{n})$$

was considered by them, where $T_\varkappa(\mathfrak{n})$ is the number of ways of expressing integral ideal $\mathfrak{n}$ in Ω as a product of $\varkappa$ ideal factors in K ($\varkappa\geq 2$). Its corresponding generating function is given by

$$Z(s)=\sum_{\mathfrak{n}}\frac{T_\varkappa(\mathfrak{n})}{N(\mathfrak{n})^s},\quad s=\sigma+it,$$

this Dirichlet series is absolute convergent in $\Re(s)>1$ and have the product formula

$$Z(s)=\Phi(s)\cdot Z^*(s),\quad \Phi(s)=\sum_{n=1}^{\infty}\frac{B_n}{n^s},$$

$$Z^*(s)=\prod_{i=1}^{h} L(s,\chi_i)^{\tau_i}=\sum_{n=1}^{\infty}\frac{A_n}{n^s},\quad A^*(x)=\sum_{n\leq x}A_n$$

where Dirichlet series $\Phi(s)$ is absolute convergent in $\Re(s)>\frac{1}{2}$. K^* is a Galois extension of K with Galois group $\mathfrak{G}$; $\chi_1, \cdots\cdots, \chi_h$ are primitive characters of $\mathfrak{G}$ and χ_1 is principal character. $L(s, \chi_i)$ are Artin's L-functions. Then we have

$$A(x)=\sum_{nn'\leq x} B_n A_{n'}$$

and their

Hauptsatz :

$$A(x)=xP_\varkappa(\log x)+\Delta_\varkappa(x)$$

and

$$P_\varkappa(\log x)=\sum_{i=1}^{\tau_1} a_i \log^{\tau_1-i-1}x,$$

where $a_0, a_1, \cdots\cdots, a_{\tau_1-1}$ are constants depending only on $\varkappa$, Ω and K and $a_0>0$.

If B_n and A_n are known, then $A(x)$ is calculated explicitly.

If K is Abelian extension of the rational number field and $l=1$ then

$$\zeta_K=\zeta_P(s)\prod_{\chi\neq\chi_1} L(s,\chi),$$

where ζ_K is zeta function of K [1]. In this case, our unknown coefficients of $P_\varkappa$ are decided explicitly by our Theorem 1 and Stirling's formula [4]. Our method is used to Iseki's extension of Hardy–Landau's identity [6].

If $l=m=1$, then

$$Z(s)=\zeta^\varkappa(s),\quad T_\varkappa(n)=d_\varkappa(n)$$

where $\zeta(s)$ is the Riemann's zeta function and $d_\varkappa(n)$ is the number of ways of expressing n as a product of $\varkappa$ factors. We have ([6], [9], and [3] Chap. X.),

$$A_\varkappa(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\zeta^\varkappa(w)\frac{x^w}{w}dw=xP_\varkappa(\log x)+\Delta_\varkappa(x)\qquad(c>1)$$

and $xP_\varkappa(\log x)$ is the residue at $w=1$. In this case, $P_\varkappa$ are given by theorem 2 explicitly.

2.

Lemma 1. We define symbolic powers s^i by means of

$$s^i=s^i(\varkappa)=\sum_{n_i=1}^{\varkappa-1}\sum_{n_{i-1}=1}^{n_i}\cdots\cdots\sum_{n_1=1}^{n_2}\sum_{n_0=1}^{n_1}1\qquad(i\geqq 1),$$

and

$$s^0=s^0(\varkappa)=\varkappa,$$

then we have

$$s^i=s^i(\varkappa)=\binom{\varkappa}{i+1},\qquad(0\leqq i\leqq\varkappa-2,\ \varkappa\geqq 2),$$

where $\binom{\varkappa}{i+1}$ are binomial coefficients.

Proof. The case $\varkappa=2$ is trivial. If we assume that the result is true for $k\leqq l$ and $0\leqq i\leqq l-2$, then it is easy to prove that our result is also true for $\varkappa\leqq l+1$ and $0\leqq i\leqq l-1$. Thus, lemma 1 is proved by induction.

Lemma 2. Put

$$f^\varkappa(s)=\sum_{i=-k}^{\infty}b^\varkappa_{\varkappa+i}(s-1)^i,\ b^\varkappa_0=1,\ b^1_i=b_i\qquad(i\geqq 0),$$

then we have

$$b^\varkappa_p=\sum_{n=0}^{p-1}s^n\sum_{\substack{(\lambda,p)\\ \lambda_1+\cdots+\lambda_p=n+1}}\gamma(\lambda_1,\cdots\cdots,\lambda_p)\,b_1^{\lambda_1}\cdots\cdots b_p^{\lambda_p}$$

$$(k\geqq 0,\ k\geqq p\geqq 1),$$

and

$$\gamma(\lambda_1,\cdots\cdots,\lambda_p)=\sum_{j=1}^{p-n}\gamma(\lambda_1,\cdots\cdots,\lambda_j-1,\ \cdots\cdots,\ \lambda_p),$$

where

$$\gamma(\lambda_1,\cdots\cdots,\lambda_p)=0\qquad\text{for some } \lambda_i<0,$$

and a partition $(1^{\lambda_1}2^{\lambda_2}3^{\lambda_3}\cdots\cdots p^{\lambda_p})$ is written for brevity as (λ,p). [8]

Proof. From the identity $f^\varkappa=f^{\varkappa-1}\cdot f$, we have

$$F^\varkappa_p=\sum_{i=0}^{p}b_{p-i}F^{\varkappa-1}_i.$$

The case $p=1$ is trivial, for $b_1^{\varkappa}=kb_1=s^0b_1$. We prove our result by induction. We assume that the result is true for all $b_q^{\varkappa}$, $q\leqq p-1$, $\varkappa\geqq 0$, $1\leqq q\leqq\varkappa$, then we have from the above

$$b_p^{\varkappa}=s^0b_p+s\sum_{i=1}^{p-1}b_{p-i}b_i^{\varkappa-1}$$

$$=s_0b_p+s\sum_{i=1}^{p-1}b_{p-i}\sum_{m=0}^{i-1}s^m\sum_{\mu_1+\cdots+\mu_i=m+1}\gamma(\mu_1,\cdots\cdots,\mu_i)\,b_i^{\mu_i}\cdots\cdots b_1^{\mu_1}$$

$$=s_0b_p+\sum_{i=1}^{p-1}\sum_{m=0}^{i-1}s^{m+1}\sum_{\mu_1+\cdots+\mu_i=m+1}\gamma(\mu_1,\cdots\cdots,\mu_i)\,b_1^{\mu_1}\cdots\cdots b_i^{\mu_i}\cdot b_{p-i}$$

$$=s^0b_p+\sum_{n=1}^{p-1}s^n\sum_{i=n}^{p-1}\sum_{\mu_1+\cdots+\mu_i=n}\gamma(\mu_1,\cdots\cdots,\mu_i)\,b_1^{\mu_1}\cdots\cdots b_i^{\mu_i}\cdot b_{p-i}$$

$$=\sum_{n=0}^{p-1}s^n\sum_{i=n}^{p-1}\sum_{\mu_1+\cdots+\mu_i=n}\gamma(\mu_1\cdots\cdots\mu_i)\,b_1^{\mu_1}\cdots\cdots b_i^{\mu_i}\cdot b_{p-i}$$

thus we obtain the relations and our lemma was proved.

Let p=6, n=2, then we have

$\gamma(011000)=\gamma(001000)+\gamma(010000)=1,$ etc.

and

$\gamma(111000)=\gamma(011000)+\gamma(101000)+\gamma(110000)+\cdots\cdots$
$=2+2+2+0+0+0=6.$

Lemma 3.

$$\lim_{m\to\infty}\left(\sum_{m=0}^{n}\frac{\log^q(m+w)}{m+w}-\frac{1}{q+1}\log^{q+1}n\right)=F_q,\qquad q\geqq 1.$$

if $w=0$, then $F_q=C_q$ are q-th Euler's constants.

Proof. See [4] Lemma 5.

Lemma 4. (Denjoy). The integral function

$$G_n(s)=H_n(s)-\zeta(s,w)$$

tends uniformly to 0 as $n\to\infty$ in the region of semi-plane $\Delta(\varepsilon)$ $(\sigma>1+\varepsilon)$ and of rectangle $R(\varepsilon,A)$ $[-1+\varepsilon<\sigma<2,\ t<A]$, ε and A are positive independent, where

$$\zeta(s,w)=\sum_{n=0}^{\infty}\frac{1}{(n+w)^s},\qquad 0<w\leqq 1,$$

is Hurwitz's zeta function [9], and this reduces to $\zeta(s)$ when $w=1$, and

$$H_n(s)=\sum_{p=1}^{n}\frac{1}{(p+w)^s}+\frac{(n+\frac{1}{2}+w)^{1-s}}{s-1}.$$

We have

$$\zeta(s,w)=\sum_{i=-1}^{\infty}F_{1+i}^{*}(s-1)^i,\quad F_0^{*}=1,$$

where

$$F_j^{*}=\frac{(-1)^{j+1}}{(j+1)!}F_{j-1}.$$

Proof. See [2].

3.

Theorem 1. In the region of lemma 4, we have

$$L(s,\chi)=\frac{E(\chi)}{k}h\frac{1}{s-1}+\Big(\frac{C_0E(\chi)}{k}h+\sum_{a=1}^{k}\frac{\chi(a)}{a}-\frac{1}{k}\sum_{a=1}^{k}\chi(a)\int_0^{\frac{a}{k}}\zeta(2,t)\,dt\Big)+g(s)(s-1),$$

where $L(s, \chi)$ are Dirichlet's L-functions mod k, $g(s)$ are integral functions of s and we write

$$E(\chi)=\begin{cases}1, & \text{for } \chi=\chi_1,\\ 0, & \text{for } \chi\neq\chi_1.\end{cases}$$

Proof. We have easily our theorem from lemma 4 and the formula ([7], Salz 368),

$$L(s,\chi)=\frac{1}{k^s}\sum_{a=1}^{k}\chi(a)\,\zeta(s,\frac{a}{k}),$$

and

$$\sum_{a=1}^{k}\chi(a)=E(\chi)h.$$

Corollary.

$$L(1)=\sum_{a=1}^{k}\frac{\chi(a)}{a}-\frac{1}{k}\sum_{a=1}^{k}\chi(a)\int_0^{\frac{a}{k}}\zeta(2,t)dt$$

Theorem 2.

$$P_\varkappa(\log x)=\sum_{i=0}^{\varkappa-1}\frac{a_i}{i\,!}\log^i x,$$

where

$$a_i=\sum_{q=0}^{\varkappa-1-i}\sum_{n=0}^{q-1}(-1)^{\varkappa-1-i-q}\binom{\varkappa}{n+1}\sum_{\lambda_1+\cdots+\lambda_q=n+1}F_1^{*\lambda_1}\cdots\cdots F_p^{*\lambda_p},$$

and

$$F_j^*=\frac{(-1)^{j+1}}{(j+1)!}F_{j+1}$$

Proof. From lemma 4, we have

$$\zeta(s)=\sum_{i=-1}^{\infty}\frac{(-1)^{i+1}}{(i+1)!}C_{i+1}(s-1)^i,$$

Thus, we obtain

$$\frac{x^s}{s}\zeta^\varkappa(s)=x\sum_{h=0}^{\infty}\sum_{i=0}^{h}(-1)^{h-i}\frac{\log^i x}{i!}(s-1)^h\times\sum_{j=0}^{\infty}b_j^k(s-1)^{j-k}$$

$$=\frac{x}{s-1}P_\varkappa(\log x)+\text{integral function},$$

where $b_j^\varkappa$ are given by lemma 2, $f(s)\equiv\zeta(s)$, and we get our theorem.

Examples.

$P_2(\log x)=\log x+2b_1-1$, (Dirichlet, 1849, [3], p. 282),

$P_3(\log x)=\frac{1}{2!}\log^2 x+(3b_1-1)\log x+(3b_2+3b_1^2-3b_1+1)$,

$P_4(\log x)=\frac{1}{3!}\log^3 x+\frac{1}{2!}(4b_1-1)\log^2 x+(6b_1^2+4b_2-4b_1+1)\log x$

$$+(4b_1^3+12b_1b_2+4b_3-6b_1^2-4b_2+4b_1-1)$$

$$P_5(\log x)=\frac{1}{4!}\log^4 x+\frac{1}{3!}(5b_1-1)\log^3 x$$

$$+\frac{1}{2!}(10b_1^2+5b_2-5b_1+1)\log^2 x$$

$$+(10b_1^3+20b_1b_2+5b_3-10b_1^2-5b_2+5b_1-1)\log x$$

$$+\{(5b_1^4+30b_1^2b_2+20b_1b_3+10b_2^2+5b_4)$$

$$-(10b_1^3+20b_1b_2+5b_3)+(10b_1^2+5b_2)-5b_1+1\},$$

where, b are given by Lemma 2, and

$$b_0=C_0,$$

$$b_1=-\frac{1}{1!}C_1,$$

$$b_2=\frac{1}{2!}C_2,$$

$$b_3=\frac{-1}{3!}C_3,$$

$$b_4=\frac{1}{4!}C_4.$$

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