The Science Reports of the Kanazawa University, Vol. III, No. 1, pp. 1 — 3, March, 1955.

On the Subregions bounded by the Level Curves of the Green's Function

By

Tôru AKAZA

(Received October 1, 1954)

The aim of this paper is to investigate the relation between the property T of a region R of the extended W-plane and the *Fuchsian* group Γ_0, when we map R conformally onto the unit circle $|x|<1$.

The property T is the extension of the certain geometric properties like convexity and starshapedness of the region and it is investigated by *Radó*[1], *Seidel*[2] and *Ford*[3].

§ 1. First of all we present the definition of the property T.

Definition. If $w'=T(w)$ is an analytic function of w, we say that a region R of the extended w-plane has the property T provided the point $w'=T(w)$ lies in R whenever w lies in R.

Let R has the property T, where T is the form

$$(1) \qquad T: \frac{w'}{w'-a}=A\frac{w}{w-a} \quad (a\neq 0,\ A\neq 1,\ 0<A\leqq 1).$$

Then *Walsh*[4] proved that, if 0 lies in R and the Green's function $G(w)$ for R with pole in 0 exists and R_γ is the domain with the condition:

$$(2) \qquad G(w)>-\log\gamma \qquad (0<\gamma<1),$$

the property T is also possessed by the subregions R_γ bounded by the level curves of the *Green's* function for R.

As the property T we consider a *one-valued, schlicht analytic* function $T(w)$ in R that satisfies;

$$(3) \qquad T(0)=0, \quad |T'(0)|\leqq 1.$$

Then R_γ has also the property T and it can be proved by *Walsh's* method.

Let R' denote the image of R under $w'=T(w)$, then by hyposesis R' is a subregion of R, that is, $R'=T(R)\subseteqq R$. The equality holds, when and only when $T(w)$ is a biuniformal transformation, which transforms R one-to-one and conformally onto itself. *This yields* $|T'(0)|=1$ *and* $R_\gamma'=R_\gamma$, *where* R_γ' *is the image of* R_γ.

The universal covering surface R^∞ of R is mapped onto the unit circle $|x|<1$ by the polymorphic function $x(w)$, so that the origin of R corresponds to the origin of $|x|<1$. Let $w(x)$ be the inverse function of $x(w)$, that is, the automorphic function and consider the function

$$(4) \qquad x'=x(T(w(x)))=f(x),$$

which is defined and analytic in $|x|<1$ by the analytic continuation. Since R has the property T, we have $|f(x)|<1$.

If $d\sigma_x$, $d\sigma_w$, $d\sigma_w'$, $d\sigma_x'$ *are the hyperbolic meusures corresponding to the line elements* dx, dw, dw', dx', *we have*

$$d\sigma_x' \leqq d\sigma_x \tag{5}$$

Then we have

$$d\sigma_w' \leqq d\sigma_w, \tag{6}$$

and the equality holds when and only when $T(w)$ *is a biuniformal transformation.* It is the special case of the principle of the hyperbolic measure[5].

§ 2. In § 1, the property T of (3) in R persists in the subregions R_T, and much more is it the case with the biuniformal transformation.

On the other hand the set G of all the biuniformal transformations forms a properly discontinuous group.

Suppose that Γ_0, F_0 are the *Fuchsian* group which belongs to $w\ (x)$ and its fundamental region. Then F_0 is mapped conformally onto the primcipal region R by $w\ (x)$.

By *Myrberg*[6] the biuniformal transformation which holds R invariant corresponds to a linear transformation LS, which holds $|x|<1$ invariant, when S moves over Γ_0.

All the transformation $L_\mu S_\nu$, which correspond to the elements of G, form a *Fuchsian* group Γ, in which Γ_0 is contained as the invariant subgroup. Then G is isomorphic with the factor group Γ/Γ_0

Now we come back to the property of G again.

If R has the property T, it obviously has the properties T^2, T^3, T^4, $\cdots$. Hence G is a cyclic group. Here, two cases may be considered according as the origin 0 is an inner point of R or an isolated boundary point of R.

1) The case where 0 is the inner point of R.

Since 0 corresponds to the origin of $|x|<1$ and $T(0)=0$, it must be the fixed point of ${}^{\forall}L\in \Gamma/\Gamma_0$. Hence L is an elliptic transformation, that is, $L(x)=Ax$, $(A=e^{i\theta})$.

If arg A is incommensurable with 2π, any point of F_0 is on the circumference whose center is the origin of $|x|<1$, and F_0 coincides with the unit circle $|x<1$. Hence R is also coincident with the inside of the circle whose center is 0.

If arg $A=2\pi/N$, where N is integer, L is the rigid rotational transformation about 0.

2) The case where R does not contain 0 and it is an isolated boundary point.

Since 0 is mapped to a point x_0 on the unit circle $|x|=1$, which is a fixed point of a parabolic transformation and any point of F_0 is in the outside of F_0 by the transfo-rmation L, it is inconsistent with the property of L.

Hence we obtain the following theorem.

Theorem 1. *The group G generated by all the biuniformal transformations* $T(w)$ *with the condition* (3) *in a region R, which has the property T, is isomorphic with the facter group* Γ/Γ_0 *which is the elliptic cyclic group.*

§ 3. Next we stsrt from the Fuchsian group.

Let Γ_0 be any *Fuchsian* group contained, as the invariant subgroup, in a *Fuchsian* group Γ of genus null of the second kind and further more the factor group Γ/Γ_0 be an elliptic cyclic group and $w(x)$ be the automorphic function, whose principal region is R, with respect to Γ_0.

Now we consider the function

$$(7) \qquad \bar{w}(x)=w(L(x)), \quad ({}^{\forall}L(x)\in\Gamma/\Gamma_0)$$

Then for ${}^{\forall}S\in\Gamma_0$, we obtain

$$(8) \qquad \bar{w}(S)=w(SL)=w(LL^{-1}SL)=w(LS')=w(L)=\bar{w}(x),$$

where $S'=L^{-1}S\,L$ belongs to Γ_0. Hence $\overline{w}(x)$ is also automorphic with respect to Γ_0. Futhermore, by (8) the principal region R' of $\bar{w}(x)$ coincides with R. The *Green's* function $G(w)$ in R exists by hyposesis of the second kind of Γ.

Consequently we obtain the following biuniformal transformation, which corresponds to $L(x)$,

$$(9) \qquad w'=w(L(x))=w(L\cdot w^{-1}(x))=T(w),$$

and it is a schlicht map of R onto itself, where $T(w)$ depends upon $L(x)$ only.[7] Without loss of generality we can assume that the fixed point of L is transformed to the origin of R.

Then we have $T(0)=0$ from (7) and (9), and 0 is contained in R. Hence we obtain the following theorem.

Theorem 2. *If the factor group Γ/Γ_0 is an elliptic cyclic group, then the principal region R of Γ_0 has the property T, where $T(w)$ is a biuniformal transformation in R, which corresponds to an element of Γ/Γ_0, and therfeore the property T persists in the subregions bounded by the level curves of the Green's function.*

References

1) T. Radó, "*Bemerkung über die konformen Abbildungen konvexer Gebiete*", Math. Ann. vol. **102**, (1930), pp. 428—429.

2) W. Seidel, "*Über die Randzuordnung bei konformen Abbildung*", Math. Ann. vol. **104**. (1931), p.204.

3) L. R. Ford, "*On the properties of regions which persist in the subregions bounded by the level curves of the Green's function*", Duke Math. Journal. vol. **1**. (1935), pp. 103—104.

4) J. L. Walsh, "*Note on the shape of level curves of Green's function,* Duke Math. Journal. vol. **20**. (1953), pp. 611—615.

5) R. Nevanlinna, "*Eindeutige Analytishe Funktionen,*" (1936) pp. 45—51.

6) P. J. Myrberg, "*Die Kapazität der singularen Menge der linearen Gruppen*", Annales Acad. Sci. Fenn. No. **10**. § 5.

7) P. J. Myrberg, loc. cit. 6).