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On the prime number theorem

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1. Introduction

All previous proofs of the Prime Number Theorem have been by transcendental arguments involving some appeal to the theory of functions of a complex variable. A few years ago, A. Selberg have achieved an elementary proof of the prime number theorem [2]. (Numbers in square brackets refer to the references at the end of this note). His proof is elementary in the sense that it uses practically no analysis, except the simplest properties of the logarithm.

In this note, we wish to treat of this theorem in the rational number field and to prove it purely arithmetically. Small latin characters except x, y denote natural numbers in general, x, y denote any rational numbers $\geqq 1$ and p, q represent the prime numbers. ε and δ are any rational numbers.

We must replace the logarithmic function with the number theoretic function $\lambda(x)$, namely

$$\lambda(x)=\sum_{n\leqq x}\frac{1}{n}, \qquad x\geqq 1.$$

And so, our prime number theorem have the following type :

$$\pi(x)\sim\frac{x}{\lambda(x)},$$

where

$$\pi(x)=\sum_{p\leqq x}1$$

is the number of primes $p\leqq x$.

Selberg's inequality which is his starting point is written as follows :

$$\sum_{p\leqq x}\lambda^2(p)+\sum_{pq\leqq x}\lambda(p)\lambda(q)=2x\lambda(x)+O(x).$$

Prof. T. Tatuzawa and Mr. K. Iseki gave in their paper [4] a simple proof for this inequality, but we wish to describe the function $\lambda(x)$ in detail, so our method is based upon the Selberg's original paper and Shapiro's funcdamental sequence of functions. I express my hearty thanks to Prof. Z. Suetuna, Mr. Y. Tanaka and Prof. T. Tatuzawa for their many kind advices and suggestion to this problem.

2. On $\lambda(x)$.

So-called Landau's symbol must be interpreted as follows. Suppose that $g(x, k)$ is a positive function of x.

(i) $$f(x, k)=O(g(x, k))$$

means that

$$|f(x, k)| < cg(x, k)$$

where c is independent of x, for all values of x in question in the rational number field.

(ii) $$f(x, k) = o(g(x, k))$$

means that

$$\frac{|f(x, k)|}{g(x, k)} < \varepsilon$$

for all sufficiently large x,

(iii) $$f(x) \sim g(x)$$

means that

$$\left|\frac{f(x)}{g(x)} - 1\right| < \varepsilon$$

for all sufficiently large x.

Definition 1.

$$[x]^{\frac{1}{k}} = \sum_{n^k \leqq x} 1$$

where k is an natural number. $[x]^{\frac{1}{k}}$ is equal to a number N which satisfy a relation

$$N^k \leqq x < (N+1)^k$$

$[x]^1 = [x]$ is the integral part of x, the largest integer which does not exceed x.

Lemma 1.

$$[[x]^{\frac{1}{k}}]^{\frac{1}{h}} = [[x]^{\frac{1}{h}}]^{\frac{1}{k}} = [x]^{\frac{1}{hk}},$$

where k and h are any natural numbers.

Lemma 2.

$$\lambda(x) + \lambda(y) = \lambda(xy) + O\left(\frac{x+y}{xy}\right), \quad x \geqq 1, \quad y \geqq 1.$$

Proof.

$$\lambda(xy) - \lambda(x) - \lambda(y) = \sum_{n=[x]+1}^{[x][y]} \frac{1}{n} - \lambda(y) + \sum_{n=[x][y]+1}^{[xy]} \frac{1}{n}$$

$$= \sum_{i=1}^{[y]-1}\left(\sum_{j=1}^{[x]} \frac{1}{i[x]+j} - \frac{1}{i}\right) - \frac{1}{[y]} + O\left(\frac{x+y}{xy}\right)$$

$$= O\left(\sum_{i=1}^{[y]-1}\left(\frac{[x]}{i[x]+j} - \frac{1}{i}\right)\right) + O\left(\frac{x+y}{xy}\right)$$

$$= O\left(\sum_{i=1}^{[y]-1} \frac{1}{i^2[x]+j}\right) + O\left(\frac{x+y}{xy}\right)$$

$$= O\left(\frac{1}{[x]} \sum_{i=1}^{[y]-1} \frac{1}{i^2}\right) + O\left(\frac{x+y}{xy}\right)$$

$$= O\left(\frac{x+y}{xy}\right).$$

Corollary

(i) $$\lambda\left(\frac{x}{y}\right) = \lambda(x) - \lambda(y) + O\left(\frac{x+y}{xy}\right).$$

(ii) $$\lambda(x^k)=k\lambda(x)+O\left(\frac{k}{x}\right).$$

(iii) $$\lambda([x]^{\frac{1}{k}})=\frac{1}{k}\lambda(x)+O(1).$$

Proof of (iii). From our definition 1, we get

$$([x]^{\frac{1}{k}})^k\leqq x<([x]^{\frac{1}{k}}+1)^k$$

$$k\lambda([x]^{\frac{1}{k}})+O(k)\leqq\lambda(x)<k\lambda([x]^{\frac{1}{k}})+O\left(\frac{k}{[x]^{\frac{1}{k}}+1}\right)+O(k)$$

$$=k\lambda([x]^{\frac{1}{k}})+O(k)$$

Lemma 3.

$$\lambda([x^k]^{\frac{1}{h}})=\frac{k}{h}\lambda(x)+O(k)$$

$$\lambda\left(\left([x]^{\frac{1}{h}}\right)^k\right)=\frac{k}{h}\lambda(x)+O(k)$$

Lemma 4.

$$\lambda(x)=O(x).$$

Proof. Suppose that ε is given sufficiently small, then a number N is determined such that if $m>N$, then $\frac{1}{m}<\varepsilon$. And so,

$$\frac{\lambda(x)}{x}<\frac{\sum_{n\leqq x}\frac{1}{n}}{[x]}=\frac{\sum_{n=1}^{N}\frac{1}{n}}{[x]}+\frac{\sum_{n=N+1}^{[x]}\frac{1}{n}}{[x]}$$

$$<\frac{\lambda(N)}{[x]}+\frac{1}{[x]}([x]-N)\varepsilon<\varepsilon'$$

Lemma 5.

$$\lambda^k(x)\equiv(\lambda(x))^k=O(x)$$

Proof. If we give any ε, we can determine N such that $\frac{1}{m}<\varepsilon$ follows from $m>N$ and take sufficiently large such as

$$\frac{\varepsilon}{2}<\frac{1}{[x]}\leqq\varepsilon,\ \text{i. e.}\ 1\leqq[x]\varepsilon<2$$

then

$$\frac{\lambda^k(x)}{x}\leqq\frac{\lambda^k(x)}{[x]}\leqq\frac{1}{[x]}(\lambda(N)-([x]-N\varepsilon))^k$$

$$=\sum_{l=0}^{k}\binom{k}{l}\lambda^l(N)2^{k-l-1}\left(1-\frac{N}{[x]}\right)^{k-l}\cdot\varepsilon$$

$$<\varepsilon'$$

Corollary.

$$\lambda^k(x)=O([x]^{\frac{1}{n}})$$

where n is any natural number.

Lemma 6.

$$\sum_{n\leq x}\frac{1}{\lambda(n)}=O\Big(\frac{x}{\lambda(x)}\Big)$$

Proof.

$$\sum_{n\leq x}\frac{1}{\lambda(n)}=\sum_{n\leq[x]^{\frac{1}{2}}}+\sum_{[x]^{\frac{1}{2}}<n\leq x}=O\Big(\frac{x}{\lambda(x)}\Big).$$

Lemma 7.

$$\sum_{n\leq x}\frac{1}{n\lambda(n)}=O(\lambda\lambda(x)).$$

where

$$\lambda\lambda(x)=\lambda(\lambda(x)).$$

Proof. We put

$$\Lambda=[\lambda(x)],\quad [x]=N\Lambda+r(x),\quad |r(x)|<\Lambda.$$

then

$$\sum_{n\leq x}\frac{1}{n\lambda(n)}=\sum_{n\leq N\Lambda}+\sum_{N\Lambda<n\leq x}=\sum\nolimits_1+\sum\nolimits_2,$$

$$\sum\nolimits_1=\sum_{l=1}^{\Lambda-1}\Big(\sum_{i=1}^{N}\frac{1}{(lN+i)\lambda(lN+i)}\Big)=O\Big(\sum_{l=1}^{\Lambda}\frac{N}{lN\lambda(lN)}\Big)=O\Big(\sum_{l=1}^{\Lambda}\frac{1}{l}\Big)$$
$$=O(\lambda\lambda(x)).$$
$$\sum\nolimits_2=O\Big(\frac{1}{(N\Lambda+1)\lambda(N\Lambda+1)}+\cdots\cdots+\frac{1}{(N\Lambda+r)\lambda(N\Lambda+r)}\Big)$$
$$=O\Big(\frac{r(x)}{N\Lambda\lambda(N\Lambda)}\Big)=O(1).$$

Lemma 8.

$$\sum_{n\leq x}\frac{\lambda(n)}{n}=\frac{1}{2}\lambda^2(x)+O(\lambda(x)).$$

Proof.

$$\sum_{n\leq x}\frac{\lambda(n)}{n}=\sum_{n\leq x}\lambda(n)(\lambda(n)-\lambda(n-1))$$
$$=-\sum_{n\leq x}\lambda(n)(\lambda(n)-\lambda(n-1))+\lambda^2(x)+O(\lambda(x))$$
$$=-\sum_{n\leq x}\frac{\lambda(n)}{n}+\lambda^2(x)+O(\lambda(x)).$$

And so we get

$$\sum_{n\leq x}\frac{\lambda(n)}{n}=\frac{1}{2}\lambda^2(x)+O(\lambda(x)).$$

Lemma 9. *We put*

$$T(x\,;\,\lambda)\equiv T(x)\equiv\sum_{n\leq x}\lambda(n),$$

then

$$T(x)=x\lambda(x)-x+O(\lambda(x))$$
$$=x\lambda(x)+O(x)$$

Proof.

$$T(x)=\sum_{n\leq x}\lambda(n)=[x]\lambda(x)-\sum_{i=1}^{[x]-1}\frac{i}{i+1}$$

$$=[x]\lambda(x)-[x]+\sum_{i=0}^{[x]-1}\left(1-\frac{i}{i+1}\right)$$

$$=x\lambda(x)-x+O(\lambda(x)).$$

Lemma 10.

$$T_2(x)=\sum_{n\leq x}\lambda^2(n)=x\lambda^2(x)-2x\lambda(x)+2x+O(\lambda^2(x)).$$

Proof.

$$T_2(x)=\sum_{n\leq x}\lambda^2(n)=\sum_{n\leq x}\lambda(n)(T(n)-T(n-1))$$

$$=-\sum_{n\leq x}\frac{T(n)}{n}+\lambda(x)T(x)+O\left(\frac{T(x)}{x}\right)$$

$$=-\sum_{n\leq x}\lambda(n)+O\left(\sum_{n\leq x}\frac{\lambda(n)}{n}\right)+x\lambda^2(x)-x\lambda(x)+O(\lambda^2(x))$$

$$=x\lambda^2(x)-2x\lambda(x)+2x+O(\lambda^2(x))$$

Lemma 11.

$$\sum_{n\leq x}\lambda^2\left(\frac{x}{n}\right)=2x+O(\lambda^2(x)).$$

Proof.

$$\sum_{n\leq x}\lambda^2\left(\frac{x}{n}\right)=\sum_{n\leq x}\left(\lambda(x)-\lambda(n)+O\left(\frac{n+x}{nx}\right)\right)^2$$

$$=\sum_{n\leq x}\lambda^2(x)+\sum_{n\leq x}\lambda^2(n)-2\lambda(x)\sum_{n\leq x}\lambda(n)+O\left(\sum_{n\leq x}\left(\frac{n+x}{nx}\right)^2\right)$$

$$+O\left(\lambda(x)\sum_{n\leq x}\frac{x+n}{nx}\right)+O\left(\sum_{n\leq x}\lambda(n)\frac{x+n}{nx}\right)$$

$$=2x+O(\lambda^2(x))$$

Lemma 12.

$$\sum_{p\leq x}\frac{\lambda(p)}{p}=\lambda(x)+O(1)$$

Proof.

$$T(x)=\sum_{n\leq x}\lambda(n)=\lambda([x]!)+O(x)$$

$$=\lambda(\prod_p p^{\sum_m[\frac{x}{p^m}]})+O(x)$$

$$=\sum_{p\leq x}\lambda(p^{\sum_m[\frac{x}{p^m}]})+O(x)+O(\pi(x))$$

$$=\sum_{p\leq x}\sum_m\left[\frac{x}{p^m}\right]\lambda(p)+O\left(\frac{1}{p}\sum_m\left[\frac{x}{p^m}\right]\right)+O(x)$$

$$=\sum_{p\leq x}\left[\frac{x}{p}\right]\lambda(p)+\sum_{p\leq x}\sum_{m\geq 2}\left[\frac{x}{p^m}\right]\lambda(p)+O(x)$$

$$=x\sum_{p\leq x}\frac{\gamma(p)}{p}+O(x)$$

From Lemma 9, we get

$$x\lambda(x)+O(x)=x\sum_{p\leq x}\frac{\lambda(p)}{p}+O(x)$$

$$\sum_{p\leq x}\frac{\lambda(p)}{p}=\lambda(x)+O(1)$$

Lemma 13.

$$\sum_{p\leq x}\frac{\lambda^2(p)}{p}=\frac{1}{2}\lambda^2(x)+O(\lambda(x)).$$

Proof. By partial summation we get this result from lemma 12. Writing now

$$R(x)=\sum_{p\leq x}\frac{\lambda(p)}{p}=\lambda(x)+O(1)$$

then

$$\sum_{p\leq x}\frac{\lambda^2(p)}{p}=\sum_{p\leq x}(R(n)-R(n-1))\lambda(n)$$

$$=-\sum_{n\leq x}R(n)(\lambda(n)-\lambda(n-1))+R(x)\lambda(x)$$

$$=-\sum_{n\leq x}\frac{\lambda(n)}{n}+O\left(\sum_{n\leq x}\frac{1}{n}\right)+\lambda^2(x)+O(\lambda(x))$$

$$=\frac{1}{2}\lambda^2(x)+O(\lambda(x)).$$

3. On $\vartheta(x, \lambda)$ and $\psi(x, \lambda)$.

We must define Tchebyschef's functions $\vartheta(x, \lambda)$ and $\psi(x, \lambda)$ in our idea namely :

Definition 2.

$$\vartheta(x, \lambda)\equiv\vartheta(x)\equiv\sum_{p\leq x}\lambda(p),$$

$$\psi(x, \lambda)\equiv\psi(x)\equiv\sum_{m=1}^{\infty}\sum_{p^m\leq x}\lambda(p)$$

Lemma 14. *$F(x)$ and $G(x)$ are two functions defined for $x\geqq 1$, which are connected by following relation*

$$G(x)=\sum_{n\leq x}F\left(\frac{x}{n}\right)+R(x),$$

where $R(x)=0$, or $O(x)$, then

$$F(x)=\sum_{n\leq x}\mu(n)G\left(\frac{x}{n}\right)+S(x),$$

where

$$S(x)=\begin{cases}0 & \text{if } R(x)=0,\\ O(x) & \text{if } R(x)=O(x),\end{cases}$$

and $\mu(n)$ means the Möbius function which is defined as follows :

(i) $\mu(1)=1$

(ii) $\mu(n)=0$, *if n has a squared factor ;*

(iii) $\mu(p_1\cdots\cdots p_r)=(-1)^r$, *if all the primes p_1, p_2, $\cdots\cdots p_r$ are different.*

Proof. (i) $R(x)=0$. In this case our result is called the Möbius inversion formula [1].

(ii) $R(x)=O(x)$.

$$\sum_{n\leq x}\mu(n)G\left(\frac{x}{n}\right)=\sum_{n\leq x}\sum_{mn}\mu(n)F\left(\frac{x}{mn}\right)+O\left(\sum_{n\leq x}\frac{x}{n}\mu(n)\right)$$

$$=\sum_{k=1}^{[x]}F\left(\frac{x}{k}\right)\sum_{n\mid k}\mu(n)+O(x)$$

$$=F(x)+O(x)$$

since we have

$$\sum_{n\mid k}\mu(n)=\begin{cases}1 & k=1,\\ 0 & k>1.\end{cases}$$

and the following lemma 15.

Now, we shall use Shapiro's fundamental sequence of functions which arise from the following inductive construction

$$f_0(x)\equiv 1$$

$$f_i(x)=\sum_{n\leq x}f_i\left(\frac{x}{n}\right)$$

for all rational number x greater than or equal 1. In our calculations we modify this construction by using a function $\bar{f}_{i-1}(x)\sim f_i(x)$ in constructing $f_i(x)$ as Shapiro do so

From the Möbius inversion formula (Lemma 14 (i)), we get

$$\bar{f}_{i-1}(x)=\sum_{n\leq x}\mu(n)f_i\left(\frac{x}{n}\right),\qquad i=1,\ 2,\ 3,\ \cdots\cdots$$

If we put $i=1$ or $i=2$, next two lemmas are given for us.

Lemma 15.

$$\sum_{n\leq x}\frac{\mu(n)}{n}=O(1).$$

Lemma 16.

$$\sum_{n\leq x}\frac{\mu(n)}{n}\lambda\left(\frac{x}{n}\right)=O(1)$$

Proof. Since,

$$f_2(x)=x\lambda(x)$$

Lemma 17.

$$\sum_{n\leq x}\psi\left(\frac{x}{n}\right)=T(x)+O(x).$$

Proof. From our proof of lemma 12

$$T(x)=\sum_{p\leq x}\sum_{m}\left[\frac{x}{p^m}\right]\lambda(p)+O(x)$$

And so, we have

$$T(x)=\sum_{m}\sum_{p^m\leq x}\lambda(p)\sum_{n=1}^{\frac{x}{p^m}}1+O(x)$$

$$=\sum_m \sum_{n\leq x} \sum_{p^m\leq \frac{x}{n}} \lambda(p)+O(x)$$

$$=\sum_{n\leq x}\psi\left(\frac{x}{n}\right)+O(x).$$

Lemma 18.

$$\psi(x)=O(x)$$

Proof. From our lemma 11, $R(x)=O(x)$, we get

$$\psi(x)=\sum_{n\leq x}\mu(n)T\left(\frac{x}{n}\right)+O(x)$$

$$=\sum_{n\leq x}\mu(n)\left(\frac{x}{n}\lambda\left(\frac{x}{n}\right)+O\left(\frac{x}{n}\right)\right)+O(x)$$

(lemma 8)

$$=x\sum_{n\leq x}\frac{\mu(n)}{n}\lambda\left(\frac{x}{n}\right)+O\left(x\sum_{n\leq x}\frac{\mu(n)}{n}\right)+O(x)$$

$$=O(x).$$

(lemma 12 and 1 3).

Lemma 19.

$$\vartheta(x)=O(x).$$

Proof.

$$\psi(x)=\sum_m \sum_{p^m\leq x}\lambda(p)=\sum_{p\leq x}\lambda(p)+\sum_{m\geq 2}\sum_{p^m\leq x}\lambda(p)$$

$$=\vartheta(x)+O(\vartheta([x]^{\frac{1}{2}})\cdot\lambda(x))$$

$$=\vartheta(x)+O(x).$$

Thus we have our proposition, since we have

$$m=O(\lambda(x))$$

where

$$2^m\leq x<3^m.$$

Lemma 20.

$$\pi(x)=O\left(\frac{x}{\lambda(x)}\right),$$

Proof.

$$\psi(x)=\sum_m \sum_{p^m\leq x}\lambda(p)$$

$$=\sum_{p\leq x}\left(\frac{\lambda(x)}{\lambda(p)}+O(1)\right)\lambda(p)$$

$$=\lambda(x)\pi(x)+O(\vartheta(x))$$

$$=\lambda(x)\pi(x)+O(x).$$

So, we have

$$\pi(x)=\frac{\psi(x)}{\lambda(x)}+O\left(\frac{x}{\lambda(x)}\right)=O\left(\frac{x}{\lambda(x)}\right).$$

Lemma 21.

$$\frac{\pi(x)}{\dfrac{x}{\lambda(x)}} \sim \frac{\vartheta(x)}{x}$$

Proof.

$$\pi(x) = \sum_{2\leq n\leq x} \frac{\vartheta(n)-\vartheta(n-1)}{\lambda(n)}$$

$$= \sum_{n\leq x} \vartheta(n)\left(\frac{1}{\lambda(n)} - \frac{1}{\lambda(n+1)}\right) + \frac{\vartheta(x)}{\lambda(x)} + O\left(\frac{1}{\lambda(x)}\right)$$

and so

$$\pi(x) - \frac{\vartheta(x)}{\lambda(x)} = O\left(\frac{x}{\lambda^2(x)}\right)$$

and

$$\sum_{n\leq x} \frac{1}{\lambda^2(n)} = O\left(\frac{x}{\lambda^2(x)}\right).$$

Our lemma follows from these relations.

Thus the prime number theorem is equivalent to the fact that

$$\vartheta(x) \sim x$$

so, it is sufficient to our problem to prove this relation.

4. Selberg's Theorem.

Lemma 22.

$$V(x,\ \lambda) \equiv V(x) \equiv \sum_{n\leq x} \frac{\mu(n)}{n} \lambda^2\left(\frac{x}{n}\right) = 2\lambda(x) + O(1).$$

Proof. We may assume following functions in Shapiro's fundamental sequence of functions:

$$\bar{f}_2 = x\lambda(x)$$

$$f_3 = x\lambda^2(x) + O(x\lambda(x)).$$

Then we get from the Möbius inversion formula :

$$x\lambda(x) = \sum_{n\leq x} \mu(n)\left(\frac{1}{2}\,\frac{x}{n}\lambda^2\left(\frac{x}{n}\right)\right) + O\left(\frac{x}{n}\lambda\left(\frac{x}{n}\right)\right)$$

$$= \frac{1}{2}\, x \sum_{n\leq x} \frac{\mu(n)}{n} \lambda^2\left(\frac{x}{n}\right) + O\left(x \sum_{n\leq x} \frac{\mu(n)}{n} \lambda\left(\frac{x}{n}\right)\right)$$

$$= \frac{1}{2} x\, V(x) + O(x)$$

and so,

$$V(x) = 2\lambda(x) + O(1).$$

Lemma 23. *We put*

$$\theta_n(x,\ \lambda) \equiv \theta_n \equiv \sum_{d\mid n} \mu(d)\lambda^2\left(\frac{x}{d}\right) = \sum_{d\mid n} \lambda_d$$

then

$$\sum_{n\leq x} \theta_n = x V(x) + O(x).$$

Proof.

$$xV(x)=x\sum_{d\le x}\frac{\lambda_d}{d},\qquad \lambda_d=\mu(d)\lambda^2\left(\frac{x}{d}\right)$$
$$=\sum_{d\le x}\lambda_d\left[\frac{x}{d}\right]+O\left(\sum_{d\le x}|\lambda_d|\right)$$
$$=\sum_{n\le x}\theta_n+O(x)$$

(lemma 10).

Lemma 24.

$$\theta_n=\begin{cases}\lambda^2(x) & n=1\\ 2\lambda(p)\lambda(x)+\lambda^2(p)+O(\lambda(x)) & n=p^\alpha\\ 2\lambda(p)\lambda(q)+O(\lambda(x)) & n=p^\alpha q^\beta\\ O(\lambda(x)) & \text{otherwise.}\end{cases}$$

Proof. It is easy to say our result for first three cases.

For $\quad n=p_1^{\alpha_1}\cdots\cdots p_N^{\alpha_N}\qquad (N>2)$,

$$\theta=\sum_{d\mid n}\mu(d)\lambda^2\left(\frac{\lambda}{d}\right)=\sum_{i=1}^{N}(-1)^i\sum_{p_1\cdots p_i}\lambda^2\left(\frac{x}{p_1\cdots\cdots p_i}\right)+\lambda^2(x)$$
$$=\sum_{i=1}^{N}(-1)^i\sum_{p_1\cdots p_i}(\lambda(x)-\lambda(p_1\cdots\cdots p_i))+O\left(\frac{x+p_1\cdots\cdots p_i}{p_1\cdots\cdots p_i x}\right)\Bigg)^2+\lambda^2(x)$$
$$=\left(\sum_{i=1}^{N}(-1)^i\binom{N}{i}+1\right)\lambda^2(x)+\sum_{i=2}^{N}(-1)^i\sum_{p_1\cdots p_i}\left(\sum_i\lambda^2(p_i)-2\sum_i\lambda(p_i)\lambda(x)\right.$$
$$\left.+2\sum_{i,j}\lambda(p_i)\lambda(p_j)\right)+O(\lambda(x))$$
$$=O(\lambda(x)).$$

Lemma 25.

$$\sum_{n\le x}\theta_n=\sum_{p\le x}\lambda^2(p)+\sum_{pq\le x}\lambda(p)\lambda(q)+O(x)$$

Proof.

$$O\Big(\sum_{\substack{p^\alpha\le x\\ \alpha>1}}\lambda^2(p)\Big)=O(\lambda(x)[x]^{\frac{1}{2}})=O(x)$$

$$O\Big(\sum_{\substack{p^\alpha q^\beta\le x\\ \alpha>1}}\lambda(p)\lambda(q)\Big)=O\Big(\lambda(x)\sum_{\substack{p^\alpha\le x\\ \alpha>1}}\lambda(p)\Big)=O(\lambda(x)[x]^{\frac{1}{2}})=O(x)$$

and

$$\sum_{p\le x}\lambda^2(p)=\lambda(x)\vartheta(x)+O(x)$$

From these relations, we get

$$O\left(\sum_{p\le x}\lambda(p)\,\lambda\left(\frac{x}{p}\right)\right)=O(x)$$

Thus our lemma is proved easily.

Theorem (*Selberg*)

$$\sum_{p\le x}\lambda^2(p)+\sum_{pq\le x}\lambda(p)\lambda(q)=2x\lambda(x)+O(x).$$

Proof. This follows from Lemmas 22, 23 and 25.

Our first step to the prime number theorem has finished. Well, our next step is oriented by Mr. A. Selberg completely[2]. Of course, Many trivial notices and modifications are necessary. For example, intervals of the type

$$(x,\ e^k x)$$

must be written as

$$(\lambda(x),\quad \lambda(x)+k),$$

and so forth.

All of these and the proof of our proposition $\vartheta(x) \sim x$ may be omitted.

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References.

[1] E. Landau : Handbuch der Lehre von der Verteilung der Primzahlen, 2 Bde, 1909.

[2] A. Selberg : An elementary proof of the prime number theorem. Ann. of Math. Vol. 50, 1949.

[3] H. N. Shapiro : On a theorem of Selberg and generalization. Ann. of Math. Vol. 51, 1950.

[4] T. Tatuzawa and K. Iseki : On Selberg's elementary proof for the prime number theorem. Proc. Japan Acad. Vol. 27, 1951. No. 7.