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On the Selberg's Inequality

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1. Introduction.

A few years ago, A. Selberg have achieved an elementary proof of the prime number theorem [3] (numbers in square brackets refer to the references at the end of this note), and his proof is based upon the following Selberg's Inequality :

$$(1) \qquad \vartheta(x)\log x+\sum_{p\leq x}\log p\ \vartheta\left(\frac{x}{p}\right)=x\log x+O(x),$$

where

$$\vartheta(x)=\sum_{p\leq x}\log p.$$

H. N. Shapiro obtained his generalization of (1) and its equivalence [4]. T. Tatuzawa and K. Iseki have done it very elegantly [5].

We shall give in this note a generalized form of (1). Our method is based upon the Selberg's original paper and Shapiro's fundamental sequence of functions. Riemann's zeta-function is not used here.

I should like to express my heartful thanks to Prof. Z. Suetuna and Prof. T. Tatuzawa.

2. Some Lemmas.

Small latin characters except x denote natural numbers, x denotes a real number $\geqq 1$ and p, p_1, p_2, …… represent the prime numbers.

Shapiro's fundamental sequence of functions :

$$f_0,\ f_1,\ f_2,\ \cdots\cdots,\ f_n,\ \cdots\cdots$$

arise from the following inductive construction :

$$f_0(x)\equiv 1,$$

$$f_i(x)=\sum_{n=1}^{x}f_{i-1}\left(\frac{x}{n}\right),\quad x\geqq 1.$$

In Shapiro's calculations, he modify this construction by using a function $\bar{f}_{i-1}(x)\sim f_{i-1}(x)$ in constructing $f_i(x)$. From this construction, we obtain the next formula

$$(2) \qquad f_{i-1}(x)=\sum_{n=1}^{x}\mu(n)f_i\left(\frac{x}{n}\right),\quad i=2,\ 3,\ \cdots\cdots$$

by the Möbius's inversion formula [1].

Lemma 1. (*Gram*).

$$V_0(x)=\sum_{n\leq x}\frac{\mu(n)}{n}=O(1).$$

Lemma 2. (*Stirling's Formula*).

$$T_1(x)=\sum_{n\leq x}\log n=x\log x-x+O(\log x)$$

Proof. [1], §16, Satz.

Lemma 3. *If* $A_i^{(k)}$ *is defined by the following initial conditions and the recurrence formula:*

$$A_0^{(1)}=A_1^{(1)}=1,$$

$$A_i^{(k+1)}=A_i^{(k)}+iA_{k-i}^{(k)}$$

$$k\geqq i\ ;\ k,\ i=1,\ 2,\ 3,\ \cdots\cdots$$

then

$$A_i^{(k)}=[k]_i=k(k-1)(k-2)\cdots\cdots(k-i+1)$$

and so

$$[k]_j[k-j]_{k-i-j}=[k]_{k-i}, \qquad [k]_k=k\,!$$

Lemma 4. *Let, in general*

$$p^n_{m_1,m_2,\cdots,n_n}=\frac{n\,!}{m_1\,!\cdots\cdots m_n\,!}$$

where

$$n=m_1+m_2+\cdots\cdots+m_n,$$

then,

$$p^n_{m_1,m_2}=\binom{n}{m_1}=\binom{n}{m_2} \qquad \textit{(binomial coefficient)},$$

and

i) $(\sum_{i=1}^{s}x_i)^n=\sum p^n_{m_1,\cdots,m_n}x_1^{m_1}\cdots\cdots x_n^{m_n},$

ii) $p^n_{m_1,m_2,\cdots,m_n}=\sum_i p^{n-1}_{m_1,\cdots,m_i-1,\cdots,m_n},$

iii) $p^n_{m_1\cdots m_n}=p^n_{m_1}p^{n-m_1}_{m_2}\ p^{n-m_1-m_2}_{m_3}\cdots\cdots.$

Lemma 5. λ_n *is a partition of* n *and if there are* m_1 *parts equal to* 1, m_2 *parts equal to* 2, m_3 *parts equal to* 3, *etc. the partition* λ_n *may be written* [2].

$$\lambda_n=(1^{m_1}2^{m_2}3^{m_3}\cdots\cdots), \qquad m_i\geqq 0,.$$

We associate a monomial

$$M(\lambda_n,x)=M(\lambda_n,x_1,\cdots\cdots,x_n)=x_1^{m_1}x_2^{m_2}\cdots\cdots x_n^{m_n}$$

with a partition λ_n.

We set

$$A_n=A_n(x)=A_n(x_1,\ x_2,\ \cdots\cdots,\ x_n)=\sum_{\lambda_n}p^m_{m_1,\cdots,m_n}M(\lambda_n,\ x)$$

where,

$$m=m_1+m_2+\cdots\cdots m_n.$$

(A_n *is equal to the Wedderburn's symbol* $\begin{Bmatrix}x_1, & \cdots\cdots, & x_n\\ m_1, & \cdots\cdots, & m_n\end{Bmatrix}$ *where* $x_1,\ x_2,\ \cdots\cdots,\ x_n$ *is only variables). Then*

$$A_n=\sum_{j=1}^{n} x_j A_{n-j}$$

Proof. We put $g(t, x)$ the generating function of A_n, and so

$$g(t, x)=\sum_{i=0}^{\infty} A_i t^i=\frac{1}{2-f(t, x)}, \quad A_0=1, \quad A_1=x_1$$

where

$$f(t, x)=\sum_{i=0}^{\infty} x_i t^i, \quad (x_0=1).$$

From this and lemma 4, the desired result follows.

Lemma 6. *Let*

$$\Gamma_j^k=k\sum_{i=j+1}^{k-1}(-1)^{k-i}\binom{k-1}{i}x_{k-i}\,\Gamma_j^{(i)}, \quad 0\leqq i\leqq k-1, \ j\leqq k-2,$$

and

$$\Gamma_0^1=0, \ \Gamma_l^1=l, \ l\geqq 2,$$

then

$$\Gamma_j^k=(-1)^{k-1-j}[k]_{k-j}B_{k-1-j},$$

where

$$B_i=A_i(y), \quad y_j=\frac{x_j}{(j-1)!} \quad j=1, 2, \cdots\cdots, n.$$

Proof. The result is proved by induction using lemma 4 and lemma 5. Suppose that our proposition is true for all k', $2\leqq k'<k$ then

$$\Gamma_j^k=k\sum_{i=j+1}^{k-1}(-1)^{k-i}\binom{k-1}{i}x_{k-i}(-1)^{i-1-j}[i]_{i-j}B_{i-1-j}$$

(inductive assumption)

$$=(-1)^{k-1-j}[k]_{k-j}\sum_{i=j+1}^{k-1}y_{k-i}\,A_{i-1-j}, \quad y_i=\frac{x_i}{(i-1)!}$$

$$=(-1)^{k-1-j}[k]_{k-j}\,B_{k-1-j}$$

(lemma 5)

Lemma 7. (*Stirling's formula*).

$$T_k(x)=\sum_{n\leqq x}\log^k n=x\sum_{i=0}^{k}(-1)^{k-i}[k]_{k-i}\log^i x+O(\log x).$$

Proof. Lemma 2 establishes the lemma in the case $k=1$. Proceeding by induction we assume it for all T_j, $1\leqq j\leqq k$. Then, by partial summation, we get

$$T_{k+1}(x)=\sum_{n\leqq x}\log^{n+1}n=\sum_{n\leqq x}\log n(T_k(n)-T_k(n-1))$$

$$=-\sum_{n\leqq x}\frac{1}{n}T_k(n)+\log x T_k(x)+O(\log^k x)$$

$$=x\sum_{i=0}^{k+1}(-1)^{k-i+1}\Big(\sum_{j=0}^{k-i}[k]_j[k-j]_{k-i-j}+[k]_{k-i+1}\Big)\log^i x+O(\log^{k+1}x).$$

$$=x\sum_{j=0}^{k+1}(-1)^{(k+1)-i}[k+1]_i \log^i x+O(\log^{k+1}x).$$

Lemma 8.

$$\sum_{n\leq x}\log^k\left(\frac{x}{n}\right)=k\,!\,x+O(\log^k x)=O(x).$$

Proof.

$$\sum_{n\leq x}\log^k\left(\frac{x}{n}\right)=\sum_{n\leq x}\sum_{m=0}^{k}(-1)^m\binom{k}{m}\log^{k-m}x\log^m n$$

$$=\sum_{m=0}^{k}(-1)^m\binom{k}{m}\sum_{l=0}^{m}(-1)^l[m]_l\log^{k-l}x+O(\log^k x)$$

$$=\sum_{l=0}^{m}(-1)^l\sum_{m=0}^{k-l}(-1)^m\binom{k-l}{m}[k]_l x\log^{k-l}x+O(\log^k x)$$

$$=k\,!\,x+O(\log^k x),$$

where

$$\sum_{i\leq m}(-1)^m\binom{k}{m}[m]_i=0 \quad i\neq k.$$

Lemma 9.

$$\sum_{2\leq n\leq x}\frac{\log^k n}{n}=\frac{1}{k+1}\log^{k+1}x+C_k+O\left(\frac{\log^k x}{x}\right),$$

where

$$C_k=\lim_{x\to\infty}\left(\sum_{1\leq n\leq x}\frac{\log^k n}{n}-\frac{1}{k+1}\log^{k+1}x\right)=Const.$$

We wish to call C_k, the k-th Euler's constant. C_0 equal to the ordinary Euler's constant.

Proof. See [1], §27, Hilfssatz.

Lemma 10.

$$\sum_{n\leq x}\frac{1}{n}\log^k\left(\frac{x}{n}\right)=\frac{1}{k+1}\log^{k+1}x+\sum_{l=0}^{k}(-1)^l\binom{k}{l}C_l\log^{k-l}x+O\left(\frac{\log^k x}{x}\right).$$

Proof. Our result is easily followed by lemma 9.

Lemma 11. *We put*

$$V_k(x)=\sum_{n\leq x}\frac{\mu(n)}{n}\log^k\left(\frac{x}{n}\right),$$

then

$$V_k(x)=k\log^{k-1}x+k\sum_{j=0}^{k-1}(-1)^{k-j}\binom{k-1}{j}C_{k-1-j}V_j(x)+O(1).$$

Proof. [4]. We proceed by induction from the case $k=0$ and the theorem proved for all k', $0\leq k'<k$. Then, since $\bar{f}_k(x)=x\log^{k-1}x$, we obtain from (2) and lemma 7,

$$f_{k+1}(x)=\sum_{n\leq x}\bar{f}_k\left(\frac{x}{n}\right)=x\sum_{n\leq x}\frac{\log^{k-1}\left(\frac{x}{n}\right)}{n}$$

$$=x\left(\frac{1}{k}x\log^{n}x+\sum_{l=0}^{k-1}(-1)^{l}\binom{k-1}{l}C_{l+1}\log^{k-1-l}x\right)+O(\log^{k-1}x)$$

$$x\log^{k-1}x=\sum_{n\leq x}\mu(n)\Big[\frac{1}{k}\,\frac{x}{n}\log^{k}\left(\frac{x}{n}\right)$$

$$+\frac{x}{n}\sum_{j=0}^{k-1}(-1)^{k-j+1}\binom{k-1}{j}C_{k-j-1}\log^{j}\left(\frac{x}{n}\right)+O\left(\log^{k-1}\left(\frac{x}{n}\right)\right)\Big]$$

and so,

$$x\log^{k-1}x=\frac{1}{k}V_{k}(x)+x\sum_{j=0}^{k-1}(-1)^{k-j+1}\binom{k-1}{j}C_{k-1-j}V_{j}(x)+O\left(\log^{k-1}\left(\frac{x}{n}\right)\right).$$

This completes the induction and the proof of the theorem.

Now, we must evaluate our function $V_k(x)$ and the next lemma 12 and lemma 13 give us the general solution of this problem, but explicitly, Theorem 1 reply for us to this.

Lemma 12. *We put*

$$f_n(x)=a_n^n g_n(x)+\sum_{i=1}^{n-1}a_i^n f_i(x)+O(1),$$

then

$$f_n(x)=\left|\begin{array}{ccccc} a_n^n g_n(x) & a_{n-1}^n & a_{n-2}^n & \cdots & a_1^n \\ a_{n-1}^{n-1}g_{n-1}(x) & 1 & a_{n-2}^{n-1} & \cdots & a_1^{n-1} \\ a_{n-2}^{n-2}g_{n-2}(x) & 0, & 1, & a_{n-3}^{n-2}\cdots & a_1^{n-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ & & & & a_1^2 \\ a_1^1 g_1(x) & 0 & \cdots & 0 & 1 \end{array}\right|^{+}+O(1).$$

where right hand symbol $\overset{+}{|}\ \overset{+}{|}$ *means a permanent* [2].

Lemma 13. *Let*

$$f_i(x)=-b_n^n g_n(x)+\sum_{i=1}^{n-1}b_i^n g_i(x)+O(1),$$

then

$$f_n(x)=\frac{1}{\prod_{i=1}^{n-1}b_i^i}\left|\begin{array}{ccccc} b_n^n g_n(x) & b_{n-1}^n & b_{n-2}^n & \cdots & b_1^n \\ f_{n-1}(x) & b_{n-1}^{n-1} & b_{n-2}^{n-1} & \cdots & b_1^{n-1} \\ f_{n-2}(x) & 0 & b_{n-3}^{n-2} & \cdots & b_1^{n-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ f_2(x) & 0 & \cdots\ 0 & b_2^2 & b_1^2 \\ f_1(x) & 0 & \cdots & 0 & b_1^1 \end{array}\right|^{+}+O(1).$$

3. Proof of the Theorems.

Theorem 1. [4]

$$V_k(x)=\sum_{n\leq x}\frac{\mu(n)}{n}\log^k\left(\frac{x}{n}\right)=\sum_{j=1}^{k-1}(-1)^{k-1-j}[k]_{k-j}B_{k-1-j}\log^j x+O(1)$$

where

$$B_i=\sum_{\lambda_i}p(\lambda_i)M(\lambda_i,\ y),\ y_j=\frac{C_j}{(j-1)\,!}\quad j=1,\ 2,\ \cdots\cdots,\ n.$$

and C_j is the j-th Euler's constant (Lemma 9).

Proof. We put

$$V_k(x)=\sum_{j=1}^{k-1}\Gamma_j^k\log^j x+O(1),\quad \Gamma_{k-1}^k=k(k\geq 2),\quad \Gamma_0^1=0.$$

then, from Lemma 11,

$$V_k(x)=k\log^{k-1}x+k\sum_{j=0}^{k-1}(-1)^{k-j}\binom{k-1}{j}C_{k-1-j}V_j(x)+O(1)$$

$$=k\log^{k-1}x+k\sum_{j=2}^{k-1}(-1)^{k-j}\binom{k-1}{j}C_{k-1-j}\sum_{i=1}^{j-1}\Gamma_i^j\log^i x+O(1)$$

and so,

$$\Gamma_i^k=k\sum_{j=i+1}(-1)^{k-j}\binom{k-1}{j}C_{k-j}\Gamma_i^j$$

$$=(-1)^{k-1-j}[k]_{k-j}B_{k-1-j}\qquad\text{(lemma 6)}$$

Lemma 14 [3]. *We define $\omega(n)$ as the number of different prime factors of n, and we put*

$$\theta_{n,\,k}=\sum_{d\mid n}\mu(d)\log^k\left(\frac{x}{d}\right),\quad \textit{where}\ n=p_1^{m1}\cdots\cdots p_{\omega(n)}^{m\omega(n)},$$

then

$$\theta_{n,\,k}=\theta_{p_1\cdots p_{\omega(n)}}=\sum_{j=0}^{k}(-1)^j\sum_{\substack{l=0\\ a_1+\cdots+a_l=j\geq\omega(n)}}^{\omega(n)-1}(-1)^l p_{a_1\cdots a_{\omega(n)}}^k\times$$

$$\sum_{p_1\cdots p_l}\ \sum_{p_1^{a_1}\cdots p_l^{a_l}\leqq x}\log^{a_1}p_1\cdots\cdots\log^{a_l}p_l\cdot\log^{k-j}x.$$

especially,

$$\omega(n)=k:\qquad \theta_{p_1\cdots p_n}=k\,!\prod_{i=1}^{k}\log p_i$$

$$\omega(n)>k\qquad \theta_{p_1\cdots p_{\omega(n)}}=0$$

Proof.

$$\theta_{n,\,k}=\sum_{d\mid n}\mu(d)\log^k\left(\frac{x}{d}\right)$$

$$=\sum_{l=0}^{\omega(n)}(-1)^l\sum_{j=0}^{k}(-1)^j\binom{k}{j}\sum_{p_1\cdots p_l}\ \sum_{\Sigma a_i=j\neq 0}p_{a_1\cdots a_l}^j\log^{a_1}p_1\cdots\cdots$$

$$\cdots\cdots\log^{a_l}p_l\log^{k-j}x$$

$$=\sum_{j=0}^{k}(-1)^j\binom{k}{j}\sum_{\substack{l=0\\ \Sigma a_i=j\neq 0}}^{\omega(n)}(-1)^l p_{a_1\cdots a_l}^j\sum_{p_1\cdots p_l}\log^{a_1}p_1\cdots\cdots\log^{a_l}p_l\log^{k-j}x$$

Examining the coefficients of $\log^{a_1} p_1 \cdots\cdots \log^{a_l} p_l$, we can get easily the following result

$$j<\omega(n), \quad \sum_{l=m}^{N}(-1)^N \frac{1}{N}\binom{l}{m}\binom{N}{l}=0,$$

$$j=\omega(n), \quad 0 \leqq l<\omega(n)$$

$$0$$

$$l=\omega(n)$$

$$(-1)^{\omega(n)} p_{a_1 \cdots a_{\omega(n)}}^{\omega(n)}$$

$$j>\omega(n), \quad 0 \leqq l<\omega(n)$$

$$0$$

$$l \geqq \omega(n) \quad (-1)^{\omega(n)} p_{a_1 \cdots a_{\omega(n)}}^{j}.$$

Hence the lemma is proved.

Lemma 15.

$$\sum_{n \leqq x} \theta_n=\sum_{j=0}^{k}(-1)^j \sum_{\substack{l=0 \\ \Sigma a i=j>\omega(n)}}^{\omega(n)-1}(-1)^l p_{a_1 \cdots a_l}^k \sum_{p_1, \cdots, p_l} \theta(a_1, \cdots, a_l ; a_1, \cdots, a_l ; x) \log^{k-j} x$$

where

$$\theta(a_1, \cdots, a_l ; a_1, \cdots, a_l ; x)=\sum_{p_1^{a_1} \cdots p_l^{a_l} \leqq x} \log^{a_1} p_1 \cdots\cdots \log^{a_l} p_l$$

Proof.

$$\sum_{n \leqq x} \theta_n=\sum_{j=0}^{k} \sum_{p_1^{a_1} \cdots\cdots p_j^{a_j} \leqq x} \theta_{p_1 \cdots p_j}$$

and so the theorem follows immediately from Lemma 14.

Theorem 2. (*Selberg's formula*)

$$\sum_{n \leqq x} \theta_{n, k}=x V_k(x)+O(x)$$

where Theorem 1 *and Lemma* 15 *give the right and left hand side respectively.*

Proof. We put

$$\lambda_{d, k}=\mu(d) \log^k\left(\frac{x}{d}\right)$$

then

$$x V_k(x)=x \sum_{d \leqq x} \frac{\lambda_{d, k}}{d},$$

and the Theorem is deduced easily by the method of Selberg [3].

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