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Note on the Order of Free Distributive Lattices

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1. The problem, proposed by Dedekind, to enumerate the order $f(n)$ of the free distributive lattice $FD(n)$ with n generators seems difficult and only fragmentary results were obtained, whereas the lattice-theoretical version of the problem was completely solved by Th. Skolem, who showed that $FD(n)$ is isomorphic with 2^{2^n}. The numbers $f(n)$ have been computed for n up to six, and Morgan Ward who found $f(6)$ also discovered an asymptotic relation $\log_2\log_2 f(n) \sim n$. But it seems that no divisibility properties of these numbers are known.

The object of this little note is to prove that *$f(n)$ is even if n is even.* This is the simplest of the arithmetics of $f(n)$, which in the whole appears complicated. For instance we know that $f(1)\equiv f(5)\equiv 1$ and $f(3)\equiv 0$, but it can be shown that $f(7)\equiv f(9)\equiv 0$. $f(11)\equiv 1$ (mod. 2) ; denying the susceptible periodicity of $f(n)$ modulo 2.

2. To prove our assertion it is convenient to deal with some property of partly ordered sets in general.

An involution (i.e. biunique mapping onto itself, with period two, which reverses implication) of a partly ordered set is called *complete* if it leaves no element fixed, and *incomplete* otherwise.

LEMMA 1. *Let P and Q be partly ordered sets and suppose that P has a complete involution δ and Q has an incomplete involution ∂. Then P^Q has a complete involution.*

PROOF. For an element $f\in P^Q$ define Δf by

$$(\Delta f)(x)=\delta f(\partial x) \qquad (x\in Q).$$

It is readily seen that Δf is isotone, i.e., $\Delta f\in P^Q$, and that $f\to\Delta f$ is an involution of P^Q. But this Δ is complete, since for a fixed element y under ∂ of Q we have

$$(\Delta f)(y)=\delta f(\partial y)=\delta f(y)\neq f(y).$$

LEMMA 2. *Let P have the greatest element I and the least element O, and let Q be of finite length. Suppose moreover that P and Q have complete involutions. Then P^Q has an incomplete involution.*

PROOF. We define the involution Δ of P^Q as before. In order to show that Δ is incomplete this time, we construct a $g \in P^Q$ such that $\Delta g = g$, i.e., $g(\partial x) = \delta g(x)$ for every $x \in Q$. Now define $g(x) = O$ or $= I$ if the maximal length of chains connecting x to maximal elements of Q is longer or shorter, respectively, than the maximal length of chains connecting x to minimal elements of Q. Elements of Q having the same length for these two kinds of maximal chains will appear in pairs, since ∂ changes such an element x into $\partial x \neq x$ with the same property ; and in such a pair $(x, \partial x)$ we define $g(x) = O$ for any one element x and $g(\partial x) = I$ for the other element ∂x. It is easily seen that g is isotone and invariant under the involution Δ.

3. We can now prove our proposition immediately. The discrete partly ordered set n has a complete involution if n is even. (This is the *true* definition of evenness.) Thus $\mathbf{2}^n$ has an incomplete involution and hence $\mathbf{2}^{\mathbf{2}^n} \simeq FD(n)$ has a complete involution by the two Lemmas. This means, however, that $f(n)$ is even, proving our assertion.

We virtually have proved a slightly more general proposition that the order $f(P)$ of the free distributive lattice $FD(P) \cong \mathbf{2}^{\mathbf{2}^P}$ is even, if P is *even* in the sense that P is finite and has a complete involution.

4. In case n is odd, every involution of $FD(n)$ is incomplete since it leaves the $((n+1)/2)$-th "elementary symmetric function" of the generators fixed, and $f(n)$ is even in some instances and odd in others, as we have observed.

It is conjectured that $f(n)$ would be divisible by every prime divisor of $n+2$, revealing a semi-periodicity of period p, modulo any prime number p, of which our proposition is the simplest special case.

References

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